# MATHEMATICS OF DESIGN AND ANALYSIS OF EXPERIMENTS

# MATHEMATICS OF DESIGN AND ANALYSIS OF EXPERIMENTS

M. C. CHAKRABARTI



ASIA PUBLISHING HOUSE
BOMBAY. CALCUTTA. NEW DELHI. MADRAS
LONDON. NEW YORK

PRINTED IN INDIA

BY V. G. MOGHE AT BOMBAY UNIVERSITY PRESS, BOMBAY

AND PUBLISHED BY P. S. JAYASINGHE,

ASIA PUBLISHING HOUSE, BOMBAY 1.

# PREFACE

DURING the session 1954-55, I was invited by the authorities of the Maharaja Sayajirao University of Baroda to deliver a course of six lectures on some selected topics in Design and Analysis of Experiments for the benefit of the post-graduate students in Statistics. The invitation was repeated during the session 1955-56 also. The present book covers more or less the substance of these twelve lectures. I have received numerous requests from friends and students for copies of these lectures in the recent past and this demand has encouraged me to print these lectures in the present form.

Design of Experiments forms a fascinating branch of Statistics and though primarily it originated from agricultural experiments, it is finding more and more applications in various other fields. In India Dr. R. C. Bose started research in the mathematical construction of designs in 1938 and he has influenced a large number of workers in India and America. I had the privilege of attending some of his lectures in 1943. Most Universities offer advanced courses in Design and Analysis of Experiments and it is hoped that this book will be found useful to the students. I have not included any experimental material for two reasons. Firstly, I have not personally conducted experiments in any experimental station. Secondly, there are many excellent books dealing with this aspect. While teaching the subject, I have felt the need for solving a large number of problems which only can give a deeper insight into things. Unfortunately such a collection of problems is not easily accessible to the students. Kendall in his book, *Exercises in Theoretical Statistics* (Charles Griffin & Co. Ltd., 1954), has removed a long-felt want of students but he has also not included exercises on experimental designs as he had 'to draw a line somewhere.' The present book contains more than 150 problems and I believe that a genuine attempt at solving the problems will give the students a firmer grasp over the subject.

No originality is claimed except perhaps in the presentation of the material. I am deeply grateful to numerous friends and students for encouraging me to undertake this work. I shall greatly appreciate if any reader will bring to my notice mistakes, misprints or any defect in the book.

M. C. CHAKRABARTI

Bombay
15 July 1962

CONTENTS

CHAPTER I

# THEORY OF LINEAR ESTIMATION

1. *Notations.* Throughout this book, letters in thick-faced type will stand for matrices. In order to economise space, the single column matrix.

$$\begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_p \end{bmatrix}$$

will be denoted by $\{a_1, a_2, \ldots, a_p\}$. $\mathbf{I}_p$ will be used to denote the identity matrix of order $p$. The $p \times q$ null matrix will be denoted by $\mathbf{O}_{pq}$ and when no confusion is likely, by $\mathbf{O}$. The $p \times q$ matrix all of whose elements are unity will be denoted by $\mathbf{E}_{pq}$. Clearly

$$\mathbf{E}_{pq} \times \mathbf{E}_{qs} = q\,\mathbf{E}_{ps} \tag{1.1.1}$$

The square matrix of order $p$ having $a_1, a_2, \ldots, a_p$ along the principal diagonal and zero elsewhere will be denoted by diag $(a_1, a_2, \ldots, a_p)$, The letter $E$ will stand for mathematical expectation and matrix $\mathbf{V}$ will stand for a dispersion matrix.

2. *Estimable Parametric Function and Condition for Estimability.* Let $y_1$, $y_2, \ldots, y_n$ be $n$ independent stochastic variates having a common variance $\sigma^2$ and expectations given by

$$E(\mathbf{y}) = E\begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \cdots \\ \theta_m \end{bmatrix} = \mathbf{A}\boldsymbol{\theta} \tag{1.2.1}$$

where the $a_{ij}$'s are known constants and $\theta_1, \theta_2, \ldots, \theta_m$ are unknown parameters. A linear function

$$\mathbf{b}'\boldsymbol{\theta} = b_1\theta_1 + b_2\theta_2 + \ldots + b_m\theta_m \tag{1.2.2}$$

is said to be estimable if there is a linear function

$$\mathbf{c}'\,\mathbf{y} = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n \tag{1.2.3}$$

such that

$$E(\mathbf{c}'\,\mathbf{y}) = \mathbf{b}'\boldsymbol{\theta} \tag{1.2.4}$$

irrespective of the values of the parameters. Evidently then,

$$\mathbf{c}'\,\mathrm{A} = \mathbf{b}' \text{ or } \mathrm{A}'\,\mathbf{c} = \mathbf{b} \tag{1.2.5}$$

and a necessary and sufficient condition for $\mathbf{b}'\,\boldsymbol{\theta}$ to be estimable is that

$$\text{Rank } \mathrm{A}' = \text{Rank } (\mathrm{A}', \mathbf{b}) \tag{1.2.6}$$

(1.2.6) is equivalent to

$$\text{Rank } \mathrm{A}'\mathrm{A} = \text{Rank } (\mathrm{A}'\mathrm{A}, \mathbf{b}) \tag{1.2.7}$$

For, if (1.2.6) holds

$$\text{Rank } \mathrm{A}'\mathrm{A} \leqslant \text{Rank } (\mathrm{A}'\mathrm{A}, \mathbf{b}) = \text{Rank } (\mathrm{A}', \mathbf{b}) \begin{bmatrix} \mathrm{A} & \mathrm{O}_{n1} \\ \mathrm{O}_{1m} & \mathrm{I}_1 \end{bmatrix} \leqslant \text{Rank } (\mathrm{A}', \mathbf{b})$$
$$= \text{Rank } \mathrm{A}'\mathrm{A}$$

Hence (1.2.7). On the other hand, if (1.2.7) holds, we can find $\mathbf{x} = \{x_1, x_2, \ldots, x_m\}$ such that $(\mathrm{A}'\mathrm{A})\ \mathbf{x} = \mathbf{b}$. Writing $\mathbf{z} = \mathrm{A}\ \mathbf{x}$, we can see that $\mathrm{A}'\mathbf{z} = \mathbf{b}$ is solvable and hence (1.2.6). Oftentimes (1.2.7) is easier of application than (1.2.6).

When $\mathbf{b}'\boldsymbol{\theta}$ is estimable, we can find a unique linear estimator (1.2.3) which (*i*) is unbiased and (*ii*) possesses the least variance among unbiased estimators. This will be called the best unbiased linear estimator of the estimable linear parametric function (1.2.2). Since (1.2.2) is estimable we have at least one linear function (1.2.3) satisfying (1.2.4) and hence (1.2.5). This leads to (1.2.6) and (1.2.7). Now $E\ (\mathbf{c}'\mathbf{y} - \mathbf{b}'\boldsymbol{\theta})^2 = \sigma^2\ \mathbf{c}'\mathbf{c}$. Hence we have to minimise $\mathbf{c}'\mathbf{c}$ subject to (1.2.5). Let $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ denote Lagrangian multipliers; differentiating $\mathbf{c}'\mathbf{c} - 2\boldsymbol{\lambda}'(\mathrm{A}'\mathbf{c} - \mathbf{b})$ with respect to $\mathbf{c}$, we get

$$\mathbf{c} = \mathrm{A}\,\boldsymbol{\lambda} \tag{1.2.8}$$

$$\mathbf{b} = \mathrm{A}'\mathrm{A}\,\boldsymbol{\lambda} \tag{1.2.9}$$

On account of (1.2.7), we can find $\boldsymbol{\lambda}$ from (1.2.9) and hence $\mathbf{c}$ from (1.2.8). If $\mathrm{A}'\mathrm{A}$ is singular (1.2.9) will not give a unique $\boldsymbol{\lambda}$ but no matter which solution of (1.2.9) we take $\mathbf{c}$ is the same. Let $\boldsymbol{\lambda}^{(1)}$ and $\boldsymbol{\lambda}^{(2)}$ be two distinct solutions of (1.2.9) and let $\mathbf{c}^{(1)} = \mathrm{A}\,\boldsymbol{\lambda}^{(1)}$, $\mathbf{c}^{(2)} = \mathrm{A}\,\boldsymbol{\lambda}^{(2)}$. Then

$$\mathrm{O} = \mathrm{AA}\,(\boldsymbol{\lambda}^{(1)} - \boldsymbol{\lambda}^{(2)})$$
$$(\mathbf{c}^{(1)} - \mathbf{c}^{(2)})'\,(\mathbf{c}^{(1)} - \mathbf{c}^{(2)}) = (\boldsymbol{\lambda}^{(1)} - \boldsymbol{\lambda}^{(2)})'\ \mathrm{A}'\mathrm{A}\,(\boldsymbol{\lambda}^{(1)} - \boldsymbol{\lambda}^{(2)}) = \mathrm{O}$$
$$\therefore\ \mathbf{c}^{(1)} = \mathbf{c}^{(2)}$$

That the minimum is actually obtained can be demonstrated in the following manner. Let us assume for the sake of simplicity that the first $r$ columns of A are independent, the rest depending on them and $\mathrm{A}_1$ is the matrix of these $r$ columns and $\mathrm{A}_2$ the matrix of the remaining columns. Then $\mathrm{A} = (\mathrm{A}_1, \mathrm{A}_2)$. Let $\mathbf{b}_1 = \{b_1, b_2, \ldots, b_r\}$. A solution of (1.2.9) is

$$\boldsymbol{\lambda} = (\mathrm{A}'_1\,\mathrm{A}_1)^{-1} \begin{bmatrix} \mathbf{b}_1 \\ \mathrm{O}_{m-r,\,1} \end{bmatrix}.$$

Hence (1.2.8) can be written as

$$\mathbf{c} = (\mathbf{A}_1, \mathbf{A}_2)\,(\mathbf{A}'_1\,\mathbf{A}_1)^{-1} \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{O} \end{bmatrix} = \mathbf{A}_1\,(\mathbf{A}'_1\mathbf{A}_1)^{-1}\,\mathbf{b}_1 \ldots \qquad (1.2.10)$$

and $\mathbf{c}'\mathbf{c} = \mathbf{b}'_1\,(\mathbf{A}'_1\mathbf{A}_1)^{-1}\,\mathbf{b}_1$ (1.2.11)

Now for every $\mathbf{c}$ satisfying (1.2.5)

$$\begin{aligned}\mathbf{c}'\mathbf{c} &= \mathbf{c}'\,[\mathbf{I} - \mathbf{A}_1\,(\mathbf{A}'_1\,\mathbf{A}_1)^{-1}\mathbf{A}'_1]\,\mathbf{c} + \mathbf{b}'_1\,(\mathbf{A}'_1\mathbf{A}_1)^{-1}\,\mathbf{b}_1 \\ &= \mathbf{c}'\,[\mathbf{I} - \mathbf{A}_1\,(\mathbf{A}'_1\mathbf{A}_1)^{-1}\,\mathbf{A}'_1]'\,[\mathbf{I} - \mathbf{A}_1\,(\mathbf{A}'_1\mathbf{A}_1)^{-1}\,\mathbf{A}'_1]\,\mathbf{c} + \mathbf{b}'_1\,(\mathbf{A}'_1\mathbf{A}_1)^{-1}\,\mathbf{b}_1 \\ &\leqslant \mathbf{b}'_1\,(\mathbf{A}'_1\mathbf{A}_1)^{-1}\,\mathbf{b}_1.\end{aligned}$$

Hence the minimum is achieved for $\mathbf{c}$ given by (1.2.8).

In the general case where $i_1$th, $i_2$th, ...$i_r$th columns of $\mathbf{A}$ are independent, the rest depending on them and $\mathbf{A}_1$ is the matrix of these independent columns, $\mathbf{b}_1 = \{b_{i_1}, b_{i_2}, \ldots, b_{i_r}\}$, $\mathbf{c}$ and $\mathbf{c}'\mathbf{c}$ are given by (1.2.10) and (1.2.11). Also it is evident that $b_{i_1}, b_{i_2}, \ldots, b_{i_r}$ can be chosen arbitrarily and any other constituent of $\mathbf{b}$, say $b_j$, in order that (1.2.2) may be estimable, must be linearly related to $b_{i_1}, b_{i_2}, \ldots, b_{i_r}$ in the same way as $j$th column of $\mathbf{A}$ is related to the $i_1$th, $i_2$th, ..., $i_r$th columns.

A linear function

$$\mathbf{l}'\,\mathbf{y} = l_1 y_1 + l_2 y_2 + \ldots + l_n y_n \qquad (1.2.12)$$

will be said to belong to error if

$$E\,(l'\,\mathbf{y}) = \mathbf{O} \qquad (1.2.13)$$

irrespective of the values of $\theta_1, \theta_2, \ldots, \theta_m$. A necessary and sufficient condition for this is

$$\mathbf{A}'\,\mathbf{l} = \mathbf{O} \qquad (1.2.14)$$

If $\boldsymbol{\alpha}_1\ \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_m$ denote the column vectors of $\mathbf{A}$, by (1.2.8)

$$\mathbf{c} = \lambda_1\,\boldsymbol{\alpha}_1 + \lambda_2\,\boldsymbol{\alpha}_2 + \ldots + \lambda_m\,\boldsymbol{\alpha}_m \text{ where } \boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$$

is given by (1.2.9). Thus $\mathbf{c}$ lies in the vector space generated by $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_m$ which is also called Estimation Space. The vector $\mathbf{l}$ correspondidg to a linear function $\mathbf{l}'\,\mathbf{y}$ belonging to error is by (1.2.14) orthogonal to the estimation space. The vector space orthogonal to the estimation space is called the Error Space.

3. *Method of Least Squares and Markoff's Theorem.* As estimate of $\boldsymbol{\theta}$ we usually take $\hat{\boldsymbol{\theta}}$ which minimises

$$(\mathbf{y} - \mathbf{A}\,\boldsymbol{\theta})'\,(\mathbf{y} - \mathbf{A}\,\boldsymbol{\theta}) \qquad (1.3.1)$$

Differentiating (1.3.1) with respect to $\boldsymbol{\theta}$ and equating the derivative to zero, we get the normal equations

$$\mathbf{A}'\,\mathbf{y} = \mathbf{A}'\mathbf{A}\,\hat{\boldsymbol{\theta}} \qquad (1.3.2)$$

The above system of equations is always solvable for Rank $A' =$ Rank $(A', A'y)$ and hence Rank $(A'A) =$ Rank $(A'A, A'y)$. The solution is unique if $A'A$ is non-singular. Let $\mathbf{b}'\boldsymbol{\theta}$ be an estimable parametric function, its best unbiased linear estimate is from (1.2.8), (1.3.2) and (1.2.9)

$$c'y = \boldsymbol{\lambda}' A'y = \boldsymbol{\lambda}'A' A\hat{\boldsymbol{\theta}} = b'\hat{\boldsymbol{\theta}} \tag{1.3.3}$$

This provides an easy method of obtaining the best unbiased linear estimate of an estimable linear function of the parameters. It is also evident that

(*i*) $E(A'y) = A'A\boldsymbol{\theta}$ (1.3.4)

(*ii*) $V(A'y) = E[A'(y - A\boldsymbol{\theta})(y - A\boldsymbol{\theta})'A] = \sigma^2 A'A$ (1.3.5)

(*iii*) $c'y = \boldsymbol{\lambda}'(A'y)$, i.e. the best unbiased linear estimate of an estimable linear function of the parameters is a linear function of the elements of $A'y$.

(*iv*) If in (1.3.2), we can formally write $\hat{\boldsymbol{\theta}} = \mathbf{D}A'y$, where $\mathbf{D}$ is a square matrix of order $m$, then in (1.2.9), we can write $\boldsymbol{\lambda} = \mathbf{D}b$ and the variance of $b\hat{\boldsymbol{\theta}}$, the best estimate of the estimable linear function $\mathbf{b}'\boldsymbol{\theta}$, is $\sigma^2\boldsymbol{\lambda}'b = \sigma^2 b'\mathbf{D}'b$. One way of choosing $\mathbf{D}$ is to take it equal to

$$\begin{bmatrix} (A'_1A_1)^{-1} & O \\ O & O \end{bmatrix}$$

(*v*) When $A'A$ is singular, $\hat{\boldsymbol{\theta}}$ is not unique. But no matter which solution of (1.3.2) we take $A\hat{\boldsymbol{\theta}}$ remains the same; just as in (1.2.10), we can thus take

$$A\hat{\boldsymbol{\theta}} = A_1(A'_1A_1)^{-1}A'_1y \tag{1.3.6}$$

We can find a non-singular matrix $\mathbf{G}$ of order $r$ such that

$$(A'_1A_1)^{-1} = G'G \tag{1.3.7}$$

Hence

$$\begin{aligned} \text{Rank } A_1(A'_1A_1)^{-1}A'_1 &= \text{Rank }(A_1G'GA'_1) = \text{Rank }(GA'_1) \\ &= \text{Rank } A'_1 = r \end{aligned} \tag{1.3.8}$$

$$\therefore \quad \text{Rank }[I - A_1(A'_1A_1)^{-1}A'_1] = n - r \tag{1.3.9}$$

Also $I - A_1(A'_1A_1)^{-1}A'_1$ is symmetric and idempotent and hence its non-zero eigenvalues are all $+1$. Hence

$$\begin{aligned} E[y - A\hat{\boldsymbol{\theta}})'(y - A\hat{\boldsymbol{\theta}})] &= E(y - A\boldsymbol{\theta})'[I - A_1(A'_1A_1)^{-1}A'_1](y - A\boldsymbol{\theta}) \\ &= (n - r)\sigma^2 \end{aligned} \tag{1.3.10}$$

As

$$E(y'y) = n\sigma^2 + \boldsymbol{\theta}'A'A\boldsymbol{\theta}$$

and

$$y'y = (y - A\hat{\boldsymbol{\theta}})'(y - A\hat{\boldsymbol{\theta}}) + \hat{\boldsymbol{\theta}}A'A\hat{\boldsymbol{\theta}}$$

$$E(\hat{\boldsymbol{\theta}}'A'A\hat{\boldsymbol{\theta}}) = r\sigma^2 + \boldsymbol{\theta}'A'A\boldsymbol{\theta} \tag{1.3.11}$$

(1.3.3) and (1.3.10) constitute Markoff's Theorem, though Markoff's result is slightly less general than what we have proved above.

The expressions

$$\hat{\theta}' A' A \hat{\theta} \tag{1.3.12}$$

$$(y - A\hat{\theta})' (y - A\hat{\theta}) \tag{1.3.13}$$

are usually known respectively as the 'sum of squares due to regression when $\theta_1, \theta_2, \ldots, \theta_m$ are fitted' and the 'residual sum of squares'.

4. *Tests of Hypothesis regarding Parameters.* We now assume that $y_1, y_2, \ldots, y_n$ are independent normal variates with a common standard deviation $\sigma$ and expectations given by (1.2.1). Now

$$\begin{aligned}(y - A\theta)' (y - A\theta) &= (y - A\hat{\theta})' (y - A\hat{\theta}) + (\hat{\theta} - \theta)' A' A (\hat{\theta} - \theta) \\ &= (y - A\theta)' [I - A_1 (A'_1 A_1)^{-1} A'_1] (y - A\theta) + \\ &\quad (y - A\theta)' A_1 (A'_1 A_1)^{-1} A'_1 (y - A\theta)\end{aligned} \tag{1.4.1}$$

Remembering (1.3.8) and (1.3.9) and by Cochran's rule the first and second constituents on the right hand side are distributed independently as $\chi^2 \sigma^2$ with $n-r$ and $r$ drgrees of freedom respectively. Hence for testing the hypothesis $\theta = O$, we may apply the $F$-test where

$$\begin{aligned}F &= \frac{y' A_1 (A'_1 A)^{-1} A'_1 y/r}{(y - A\hat{\theta})' (y - A\hat{\theta})/(n-r)} \\ &= \frac{\hat{\theta}' A' y / r}{(y' y - \hat{\theta}' A' y) / (n-r)}\end{aligned} \tag{1.4.2}$$

with d. f. $r$ and $n-r$. $\hat{\theta}' A' y$ can be easily obtained by pre-multiplying the left hand side of (1.3.2) by $\hat{\theta}'$. Let

$$z = \{z_1, z_2, \ldots, z_n\}; z_1 = \{z_1, z_2, \ldots, z_r\}; z_2 = \{z_{r+1}, z_{r+2}, \ldots, z_n\}.$$

The rows of $G A'_1$ where $G$ is given by (1.3.7) are orthonormal. For

$$G A'_1 (G A'_1)' = G A'_1 A_1 G' = G (A'_1 A_1) (A'_1 A)^{-1} G^{-1} = G G^{-1} = I$$

Hence we can find further $n-r$ orthonormal vectors $L$, in an infinity of ways, such that $\begin{bmatrix} G A'_1 \\ L \end{bmatrix}$ forms a complete orthonormal system. Writing

$$z = \begin{bmatrix} G A'_1 \\ L \end{bmatrix} (y - A\theta)$$

$$z' z = (y - A\theta)' (y - A\theta)$$

$$\begin{aligned}(y - A\theta)' A_1 (A'_1 A_1)^{-1} A'_1 (y - A\theta) &= [G A'_1 (y - A\theta)]' [G A'_1 (y - A\theta)] \\ &= z'_1 z_1 = z_1^2 + z_2^2 + \ldots + z_r^2\end{aligned} \tag{1.4.3}$$

$$(\mathbf{y}-\mathbf{A}\,\boldsymbol{\theta})'\,[\mathbf{I}-\mathbf{A}_1\,(\mathbf{A}'_1\,\mathbf{A}_1)^{-1}\,\mathbf{A}'_1]\,(\mathbf{y}-\mathbf{A}\,\boldsymbol{\theta})=\mathbf{z}'_2\,\mathbf{z}_2=z^2_{r+1}+z^2_{r+2}+\ldots+z^2_n \tag{1.4.4}$$

$z_1, z_2, \ldots, z_n$ are independent normal with expectations zero and variance $\sigma^2$. From (1.2.10), the best unbiased linear estimate of the estimable parametric function $\mathbf{b}'\,\boldsymbol{\theta}$ is $\mathbf{b}'_1\,(\mathbf{A}'_1\,\mathbf{A}_1)^{-1}\,\mathbf{A}'_1\,\mathbf{y}=\mathbf{b}'\,\boldsymbol{\theta}+\mathbf{b}'_1\,\mathbf{G}'\,\,\mathbf{G}\,\mathbf{A}'_1\,(\mathbf{y}-\mathbf{A}\,\boldsymbol{\theta})=\mathbf{b}'\,\boldsymbol{\theta}+\mathbf{b}'_1\,\mathbf{G}'\,\mathbf{z}_1$ and depends on $z_1, z_2, \ldots, z_r$ and not on $z_{r+1}, \ldots, z_n$. Hence to test $\mathbf{b}'\,\boldsymbol{\theta}=\mathbf{O}$, we may apply $t$-test where

$$t=\frac{\mathbf{b}'_1\,(\mathbf{A}'_1\,\mathbf{A}_1)^{-1}\,\mathbf{A}'_1\,\mathbf{y}/\sqrt{\mathbf{b}'_1\,(\mathbf{A}_1'\,\mathbf{A}_1)^{-1}\,\mathbf{b}_1}}{\sqrt{(\mathbf{y}'\,\mathbf{y}-\hat{\boldsymbol{\theta}}'\,\mathbf{A}'\,\mathbf{y})\,/\,(n-r)}} \tag{1.4.5}$$

with $(n-r)$ degrees of freedom.

5. *Test Involving Several Linear Functions of Parameters.* Let $\mathbf{b}'_i\,(\boldsymbol{\theta})$. $i=1, 2, \ldots, k$ be $k$ estimable parametric functions and let the first $k'$ of them be independent, the rest depending on them. Let $\mathbf{b}_i=\{b_{i1}, b_{i2}, \ldots, b_{ir}, \ldots, b_{im}\}$ and $\mathbf{b}_{i1}=\{b_{i1}, b_{i2}, \ldots, b_{ir}\}$. Then by (1.2.10) $\mathbf{b}'_{i1}\,(\mathbf{A}'_1\,\mathbf{A}_1)^{-1}\,\mathbf{A}'_1\,\mathbf{y}$ will be the best unbiased linear estimate of $\mathbf{b}'_1\,\boldsymbol{\theta}$. The dispersion matrix of the first $k'$ of these estimates is the non-singular matrix

$$\mathbf{B}_1\,(\mathbf{A}'_1\,\mathbf{A}_1)^{-1}\,\mathbf{B}'_1$$

where $\mathbf{B}_1=\{\mathbf{b}'_{11}, \mathbf{b}'_{21}, \mathbf{b}'_{k'1}\}$.

When $\mathbf{b}'_i\,\boldsymbol{\theta}=\mathbf{O}$, $i=1, 2, \ldots, k$

$$\begin{aligned}&\mathbf{y}'\,\mathbf{A}_1\,(\mathbf{A}'_1\,\mathbf{A}_1)^{-1}\,\mathbf{B}'_1\,[\mathbf{B}_1\,(\mathbf{A}'_1\,\mathbf{A}_1)^{-1}\,\mathbf{B}'_1]^{-1}\,\mathbf{B}_1\,(\mathbf{A}'_1\,\mathbf{A}_1)^{-1}\,\mathbf{A}'_1\,\mathbf{y}\\ &=(\mathbf{y}-\mathbf{A}\,\boldsymbol{\theta})'\,\mathbf{A}_1\,(\mathbf{A}'_1\mathbf{A}_1)^{-1}\,\mathbf{B}'_1\,[\mathbf{B}_1\,(\mathbf{A}'_1\mathbf{A})^{-1}\,\mathbf{B}'_1]^{-1}\,\mathbf{B}_1\,(\mathbf{A}'_1\mathbf{A}_1)^{-1}\,\mathbf{A}'_1\,(\mathbf{y}-\mathbf{A}\,\boldsymbol{\theta})\\ &=\mathbf{z}'_1\,\mathbf{G}\,\mathbf{B}'_1\,[\mathbf{B}_1\,(\mathbf{A}'_1\mathbf{A}_1)^{-1}\,\mathbf{B}'_1]^{-1}\,\mathbf{B}_1\,\mathbf{G}'\,\mathbf{z}_1\end{aligned} \tag{1.5.1}$$

is a quadratic form in $z_1, z_2, \ldots, z_r$ of rank $k'$.

$\mathbf{G}\,\mathbf{B}'_1\,[\mathbf{B}_1\,(\mathbf{A}'_1\mathbf{A}_1)^{-1}\,\mathbf{B}'_1]^{-1}\,\mathbf{B}_1\,\mathbf{G}'$ is besides indempotent. Hence (1.5.1) is distributed as $\chi^2\sigma^2$ with $k'$ degrees of freedom. Hence for testing $\mathbf{b}'_i\,\boldsymbol{\theta}=\mathbf{O}$, $i=1, 2, \ldots, k$, we can apply the $F$-test where

$$F=\frac{\mathbf{y}'\,\mathbf{A}_1\,(\mathbf{A}'_1\mathbf{A}_1)^{-1}\,\mathbf{B}'_1\,[\mathbf{B}_1\,(\mathbf{A}'_1\mathbf{A}_1)^{-1}\,\mathbf{B}'_1]^{-1}\,\mathbf{B}_1\,(\mathbf{A}'_1\mathbf{A}_1)^{-1}\,\mathbf{A}'_1\,\mathbf{y}/k'}{(\mathbf{y}'\mathbf{y}-\hat{\boldsymbol{\theta}}'\,\mathbf{A}'\mathbf{y})\,/\,n-r)} \tag{1.5.2}$$

with d.f. $k'$ and $n-r$.

Let $\mathbf{B}=\{\mathbf{b}'_{11}, \mathbf{b}'_{21}, \ldots, \mathbf{b}_{k1}\}$. The dispersion matrix of the best estimates of $\mathbf{b}'_i\,\boldsymbol{\theta}\;(i=1, 2, \ldots, k)$ is $\mathbf{B}\,(\mathbf{A}'_1\mathbf{A}_1)^{-1}\,\mathbf{B}'$. Let $\mathbf{v}=\{v_1, v_2, \ldots, v_k\}$. Then if we write down the equations

$$\mathbf{B}\,(\mathbf{A}'_1\mathbf{A}_1)^{-1}\,\mathbf{A}_1\,\mathbf{y}=\mathbf{B}\,(\mathbf{A}'_1\mathbf{A}_1)^{-1}\,\mathbf{B}'\,\mathbf{v}, \tag{1.5.3}$$

the expression $\mathbf{v}'\,\mathbf{B}\,(\mathbf{A}'_1\mathbf{A}_1)^{-1}\,\mathbf{A}'_1\,\mathbf{y}$ will have the same value no matter which solution of (1.5.3) we take. The numerator of (1.5.2) is proportional to the value of $\mathbf{v}'\,\mathbf{B}\,(\mathbf{A}'_1\mathbf{A}_1)^{-1}\,\mathbf{A}'_1\,\mathbf{y}$ corresponding to $v_{k'+1}=v_{k'+2}=\ldots=v_k=\mathbf{O}$.

Hence obtain any solution v of (1.5.3), pre-multiply the left-hand side of (1.5.3) by v′. Then (1.5.2) can be written in the form

$$F = \frac{\mathbf{v}'\,\mathbf{B}\,(\mathbf{A}'_1\mathbf{A}_1)^{-1}\,\mathbf{A}'_1\,\mathbf{y}\;/\;k'}{(\mathbf{y}'\mathbf{y} - \hat{\boldsymbol{\theta}}'\,\mathbf{A}'\mathbf{y})\;/\;(n-r)}$$

with d.f. $n_1 = k'$, $n_2 = n - r$.

6. *Tests of Subhypothesis.* Method of Fitting of Constants.

Let $\boldsymbol{\theta}_1 = \{\ \theta_1, \theta_2, \ldots, \theta_t\ \}$ and $\boldsymbol{\theta}_2 = \{\ \theta_{t+1}, \theta_{t+2}, \ldots, \theta_m\ \}$

Let

$$\mathbf{A}\,\boldsymbol{\theta} = (\,\mathbf{A}_3\,\mathbf{A}_4\,)\begin{bmatrix}\boldsymbol{\theta}_1\\ \\ \boldsymbol{\theta}_2\end{bmatrix} \text{ and let Rank } A_3 = s \leqslant t, \tag{1.6.1}$$

Let $\hat{\boldsymbol{\theta}}_1(\boldsymbol{\theta}_2)$ satisfy

$$\mathbf{A}'_3\,(\,\mathbf{y} - \mathbf{A}_4\,\boldsymbol{\theta}_2\,) = \mathbf{A}'_3\,\mathbf{A}_3\,\hat{\boldsymbol{\theta}}_1(\boldsymbol{\theta}_2) \tag{1.6,2}$$

Then

$$\mathbf{A}'_3\,\mathbf{y} = \mathbf{A}'_3\,\mathbf{A}\begin{bmatrix}\hat{\boldsymbol{\theta}}_1(\boldsymbol{\theta}_2)\\ \\ \boldsymbol{\theta}_2\end{bmatrix} \tag{1.6,3}$$

From (1.3.2)

$$\mathbf{A}'_3\,\mathbf{y} = \mathbf{A}'_3\,\mathbf{A}\begin{bmatrix}\hat{\boldsymbol{\theta}}_1\\ \\ \hat{\boldsymbol{\theta}}_2\end{bmatrix} \tag{1.6.4}$$

From (1.6.3) and (1.6.4) we get

$$\mathbf{O} = \mathbf{A}'_3\,\mathbf{A}\begin{bmatrix}\hat{\boldsymbol{\theta}}_1 - \hat{\boldsymbol{\theta}}_1(\boldsymbol{\theta}_2)\\ \hat{\boldsymbol{\theta}}_2 - \boldsymbol{\theta}_2\end{bmatrix} \tag{1.6.5}$$

Now $\quad (\,\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\,)'\ \mathbf{A}'\,\mathbf{A}\ (\,\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\,)$

$$= \left[\begin{bmatrix}\hat{\boldsymbol{\theta}}_1 - \hat{\boldsymbol{\theta}}_1(\boldsymbol{\theta}_2)\\ \hat{\boldsymbol{\theta}}_2 - \boldsymbol{\theta}_2\end{bmatrix} + \begin{bmatrix}\hat{\boldsymbol{\theta}}_1(\boldsymbol{\theta}_2) - \boldsymbol{\theta}_1\\ \mathbf{O}\end{bmatrix}\right]'\mathbf{A}'\,\mathbf{A}\left[\begin{bmatrix}\hat{\boldsymbol{\theta}}_1 - \hat{\boldsymbol{\theta}}_1(\boldsymbol{\theta}_2)\\ \hat{\boldsymbol{\theta}}_2 - \boldsymbol{\theta}_2\end{bmatrix}\right.$$

$$\left. + \begin{bmatrix}\hat{\boldsymbol{\theta}}_1(\boldsymbol{\theta}_2) - \boldsymbol{\theta}_1\\ \mathbf{O}\end{bmatrix}\right]$$

$$= \begin{bmatrix}\hat{\boldsymbol{\theta}}_1 - \hat{\boldsymbol{\theta}}_1(\boldsymbol{\theta}_2)\\ \hat{\boldsymbol{\theta}}_2 - \boldsymbol{\theta}_2\end{bmatrix}'\mathbf{A}'\,\mathbf{A}\begin{bmatrix}\hat{\boldsymbol{\theta}}_1 - \hat{\boldsymbol{\theta}}_1(\boldsymbol{\theta}_2)\\ \hat{\boldsymbol{\theta}}_2 - \boldsymbol{\theta}_2\end{bmatrix} + \begin{bmatrix}\hat{\boldsymbol{\theta}}_1(\boldsymbol{\theta}_2) - \boldsymbol{\theta}_1\\ \mathbf{O}\end{bmatrix}'\mathbf{A}'\,\mathbf{A}\begin{bmatrix}\hat{\boldsymbol{\theta}}_1(\boldsymbol{\theta}_2) - \boldsymbol{\theta}_1\\ \mathbf{O}\end{bmatrix} \tag{1.6.6}$$

From (1.4.1) and (1.6.6) we get

$$(\mathbf{y}-\mathbf{A}\boldsymbol{\theta})'(\mathbf{y}-\mathbf{A}\boldsymbol{\theta})=(\mathbf{y}-\mathbf{A}\hat{\boldsymbol{\theta}})'(\mathbf{y}-\mathbf{A}\hat{\boldsymbol{\theta}}) + \begin{bmatrix} \hat{\boldsymbol{\theta}}_1-\hat{\boldsymbol{\theta}}_1(\boldsymbol{\theta}_2) \\ \hat{\boldsymbol{\theta}}_2-\boldsymbol{\theta}_2 \end{bmatrix}' \mathbf{A}'\mathbf{A} \begin{bmatrix} \hat{\boldsymbol{\theta}}_1-\hat{\boldsymbol{\theta}}_1(\boldsymbol{\theta}_2) \\ \hat{\boldsymbol{\theta}}_2-\boldsymbol{\theta}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} \hat{\boldsymbol{\theta}}_1(\boldsymbol{\theta}_2)-\boldsymbol{\theta}_1 \\ \mathbf{O} \end{bmatrix}' \mathbf{A}'\mathbf{A} \begin{bmatrix} \hat{\boldsymbol{\theta}}_1(\boldsymbol{\theta}_2)-\boldsymbol{\theta}_1 \\ \mathbf{O} \end{bmatrix} \tag{1.6.7}$$

Let the first $s$ columns of $\mathbf{A}_3$ be independent, the rest depending on these $s$ columns and let the matrix of the first $s$ columns be denoted by $\mathbf{A}_5$. Then

$$\begin{bmatrix} \hat{\boldsymbol{\theta}}_1(\boldsymbol{\theta}_2)-\boldsymbol{\theta}_1 \\ \mathbf{O} \end{bmatrix}' \mathbf{A}'\mathbf{A} \begin{bmatrix} \hat{\boldsymbol{\theta}}_1(\boldsymbol{\theta}_2)-\boldsymbol{\theta}_1 \\ \mathbf{O} \end{bmatrix} = \left(\hat{\boldsymbol{\theta}}_1(\boldsymbol{\theta}_2)-\boldsymbol{\theta}_1\right)' \mathbf{A}'_3\mathbf{A}_3\left(\hat{\boldsymbol{\theta}}_1(\boldsymbol{\theta}_2)-\boldsymbol{\theta}_1\right)$$

$$=(\mathbf{y}-\mathbf{A}\boldsymbol{\theta})'\,\mathbf{A}_5(\mathbf{A}'_5\mathbf{A}_5)^{-1}\mathbf{A}'_5(\mathbf{y}-\mathbf{A}\boldsymbol{\theta}) \tag{1.6.8}$$

This is a quadratic form of rank $s$. The first constituent on the right hand side of (1.6.7) is a quadratic form of rank $(n-r)$. The rank of the second form is at least $r-s$. The second constituent is

$$(\mathbf{y}-\mathbf{A}\boldsymbol{\theta})'\,[\mathbf{A}_1(\mathbf{A}'_1\mathbf{A}_1)^{-1}\mathbf{A}'_1-\mathbf{A}_5(\mathbf{A}'_5\mathbf{A}_5)^{-1}\mathbf{A}'_5]\,(\mathbf{y}-\mathbf{A}\boldsymbol{\theta} \tag{1.6.9}$$

since $\mathbf{A}_1(\mathbf{A}'_1\mathbf{A}_1)^{-1}\mathbf{A}'_1-\mathbf{A}_5(\mathbf{A}'_5\mathbf{A}_5)^{-1}\mathbf{A}'_5$ is subject to $n-r+s$ independent restrictions.

$$\mathbf{A}'_5[\mathbf{A}_1(\mathbf{A}'_1\mathbf{A}_1)^{-1}\mathbf{A}'_1-\mathbf{A}_5(\mathbf{A}'_5\mathbf{A}_5)^{-1}\mathbf{A}'_5]=\mathbf{O} \tag{1.6.10}$$

$$\boldsymbol{\tau}'[\mathbf{A}_1(\mathbf{A}'_1\mathbf{A}_1)^{-1}\mathbf{A}'_1-\mathbf{A}_5(\mathbf{A}'_5\mathbf{A}_5)^{-1}\mathbf{A}'_5]=\mathbf{O} \tag{1.6.11}$$

where $\boldsymbol{\tau}=\{t_1, t_2, \ldots, t_n\}$ and $\mathbf{A}'\boldsymbol{\tau}=\mathbf{O}$, the rank of (1.6.9) is at most $(r-s)$. Hence the rank of (1.6.9) is exactly $r-s$. The constituents of (1.6.7) are, therefore, independently distributed as $\chi^2\sigma^2$ with $n-r$, $r-s$ and $s$ degree of freedom respectively. Ths first constituent is independent of $\boldsymbol{\theta}$. The second constituent depends on $\boldsymbol{\theta}_2$ and not on $\boldsymbol{\theta}_1$ and the third both on $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$. Hence for testing $\boldsymbol{\theta}_2=\boldsymbol{\theta}_2^\circ$, we apply the $F$-test where

$$F=\frac{\begin{bmatrix} \hat{\boldsymbol{\theta}}_1-\hat{\boldsymbol{\theta}}_1(\boldsymbol{\theta}_2^\circ) \\ \hat{\boldsymbol{\theta}}_2-\boldsymbol{\theta}_2^\circ \end{bmatrix}' \mathbf{A}'\mathbf{A} \begin{bmatrix} \hat{\boldsymbol{\theta}}_1-\hat{\boldsymbol{\theta}}_1(\boldsymbol{\theta}_2^\circ) \\ \hat{\boldsymbol{\theta}}_2-\boldsymbol{\theta}_2^\circ \end{bmatrix} / (r-s)}{(\mathbf{y}'\mathbf{y}-\hat{\boldsymbol{\theta}}'\mathbf{A}'\mathbf{y})/(n-r)} \tag{1.6.12}$$

with d. f. $n_1=r-s$ ; $n_2=n-r$

When $\boldsymbol{\theta}_2=\mathbf{O}$, the numerator of (1.6.12) is proportional to

$$\begin{bmatrix} \hat{\boldsymbol{\theta}}_1 \\ \hat{\boldsymbol{\theta}}_2 \end{bmatrix}' \mathbf{A}'\mathbf{A} \begin{bmatrix} \hat{\boldsymbol{\theta}}_1 \\ \hat{\boldsymbol{\theta}}_2 \end{bmatrix} - \hat{\boldsymbol{\theta}}_1'(\mathbf{O})\,\mathbf{A}'_3\mathbf{A}_3\hat{\boldsymbol{\theta}}_1(\mathbf{O})$$

$$=\hat{\boldsymbol{\theta}}'\mathbf{A}'\mathbf{y}-\hat{\boldsymbol{\theta}}_1'(\mathbf{O})\,\mathbf{A}'_3\mathbf{y} \tag{1.6.13}$$

= Sum of squares due to regression when $\theta_1, \theta_2, \ldots, \theta_m$ are fitted — Sum of squares due to regression when only $\theta_1, \theta_2 \ldots, \theta_t$ are fitted.
In this case

$$F = \frac{(\hat{\boldsymbol{\theta}}' \mathrm{A}' \mathrm{y} - \hat{\boldsymbol{\theta}}_1' (\mathrm{O})\, \mathrm{A}'_3 \mathrm{y})) / (r-s)}{(\mathrm{y}' \mathrm{y} - \hat{\boldsymbol{\theta}}' \mathrm{A}' \mathrm{y}) / (n-r)} \tag{1.6.14}$$

with d. f. $n_1 = r - s$; $n_2 = n - r$.
It is evident that no test is available when $r = s$.

7. *A Particular Test Involving Equality of Some of the Parameters.* Oftentimes, we are required to test $\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_{t+2} = \ldots = \boldsymbol{\theta}_m$. Let $A_6$ consist of a set of $r'$ independent columns of

$$\begin{bmatrix} a_{11}\, a_{12} \cdots a_{1t} & a_{1t+1} + a_{1t+2} + \cdots + a_{1m} \\ a_{21}\, a_{22} \cdots a_{2t} & a_{2t+1} + a_{2t+2} + \cdots + a_{2m} \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}\, a_{n2} \cdots a_{nt} & a_{nt+1} + a_{nt+1} + \cdots + a_{nm} \end{bmatrix} \tag{1.7.1}$$

the other columns depending on these columns. Then

$$\begin{aligned} (\mathrm{y} - \mathrm{A}\boldsymbol{\theta})' (\mathrm{y} - \mathrm{A}\boldsymbol{\theta}) &= (\mathrm{y} - \mathrm{A}\boldsymbol{\theta})' [\mathrm{I} - \mathrm{A}_1 (\mathrm{A}'_1 \mathrm{A}_1)^{-1} \mathrm{A}'_1] (\mathrm{y} - \mathrm{A}\boldsymbol{\theta}) \\ &+ (\mathrm{y} - \mathrm{A}\boldsymbol{\theta})' [\mathrm{A}_1 (\mathrm{A}'_1 \mathrm{A}_1)^{-1} \mathrm{A}'_1 - \mathrm{A}_6 (\mathrm{A}'_6 \mathrm{A}_6)^{-1} \mathrm{A}'_6] (\mathrm{y} - \mathrm{A}\boldsymbol{\theta}) \\ &+ (\mathrm{y} - \mathrm{A}\boldsymbol{\theta})' \mathrm{A}_6 (\mathrm{A}'_6 \mathrm{A}_6)^{-1} \mathrm{A}'_6 (\mathrm{y} - \mathrm{A}\boldsymbol{\theta}) \end{aligned} \tag{1.7.2}$$

The constituents on the right hand side of (1.7.2) are independently distributed as $\chi^2\, \sigma^2$ with $n - r$, $r - r'$ and $r'$ degrees of freedom respectively. When $\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_{t+2} = \cdots = \boldsymbol{\theta}_m$, the second constituent is simply

$$\mathrm{y}' [\mathrm{A}_1 (\mathrm{A}'_1 \mathrm{A}_1)^{-1} \mathrm{A}'_1 - \mathrm{A}_6 (\mathrm{A}'_6\, \mathrm{A}_6)^{-}{}_1\, \mathrm{A}'_6]\, \mathrm{y}$$

Hence the appropriate $F$-test is

$$F = \frac{(\mathrm{y}'\, \mathrm{A}_1 (\mathrm{A}'_1 \mathrm{A}_1)^{-1} \mathrm{A}'_1\, \mathrm{y} - \mathrm{y}'\, \mathrm{A}_6 (\mathrm{A}'_6 \mathrm{A}_6)^{-1} \mathrm{A}'_6\, \mathrm{y}) / (r - r')}{(\mathrm{y}'\mathrm{y} - \hat{\boldsymbol{\theta}}'\, \mathrm{A}'\, \mathrm{y}) / (n-r)} \tag{1.7.3}$$

with d. f. $n_1 = r - r'$; $n_2 = n - r$; $r'$ denoting the rank of (1.7.1).

8. *Linear Estimation with Correlated Variables.* Let V be the dispersion matrix of $\mathrm{y}$ and $E$ (y) be given by (1.2.1). We assume that V is non-singular. We can, therefore, find a non-singular H such that $\mathrm{H}'\, \mathrm{H} = \mathrm{V}^{-1}$. The case of correlated variables can be reduced to the case of independent variables by means of the transformation $\mathrm{z} = \mathrm{H}'^{-1}\, \mathrm{y}$ or $\mathrm{y} = \mathrm{H}'\, \mathrm{z}$.

We define an estimable linear parametric function $\mathbf{b}'\, \boldsymbol{\theta}$ in the same way as in section 2 and obtain a necessary and sufficient condition as the availability of a $\mathbf{c}$ such that $\mathrm{A}'\, \mathbf{c} = \mathbf{b}$. This gives (1.2.6) or the equivalent condition

(1.6.7.). The variance of the estimator $\mathbf{c}'\,\mathbf{y}$ is $\mathbf{c}'\ \mathbf{V}\ \mathbf{c}$ and in order that this may be a minimum subject to $\mathbf{A}'\,\mathbf{c} = \mathbf{b}$,

$$\mathbf{V}\,\mathbf{c} = \mathbf{A}\,\boldsymbol{\lambda} \text{ or } \mathbf{c} = \mathbf{V}^{-1}\,\mathbf{A}\,\boldsymbol{\lambda} \tag{1.8.1}$$

$$\mathbf{b} = \mathbf{A}'\,\mathbf{V}^{-1}\,\mathbf{A}\boldsymbol{\lambda} \tag{1.8.2}$$

where

$$\text{Rank }(\mathbf{A}'\mathbf{V}^{-1}\mathbf{A}) \leqslant \text{Rank }(\mathbf{A}'\,\mathbf{V}^{-1}\,\mathbf{A},\,\mathbf{b}) = \text{Rank }(\mathbf{A}',\mathbf{b}) \begin{bmatrix} \mathbf{V}^{-1}\,\mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_1 \end{bmatrix}$$

$\leqslant$ Rank $(\mathbf{A}',\,\mathbf{b})$ = Rank $(\mathbf{A}')$ = Rank $(\mathbf{A}'\,\mathbf{H}')$ = Rank $(\mathbf{H}\,\mathbf{A})'\,(\mathbf{H}\,\mathbf{A})$ = Rank $(\mathbf{A}'\,\mathbf{V}^{-1}\,\mathbf{A})$

$\therefore$ Rank $(\mathbf{A}'\,\mathbf{V}^{-1}\mathbf{A})$ = Rank $(\mathbf{A}'\,\mathbf{V}^{-1}\,\mathbf{A},\,\mathbf{b})$

and (1.8.2) is solvable. No matter which solution of (1.8.2), we take $\mathbf{c}$ in (1.8.1) is unique. For let $\boldsymbol{\lambda}^{(1)}$ and $\boldsymbol{\lambda}^{(2)}$ be distinct solutions of (1.8.2). Then

$$\mathbf{O} = \mathbf{A}'\,\mathbf{V}^{-1}\,\mathbf{A}\,(\boldsymbol{\lambda}^{(1)} - \boldsymbol{\lambda}^{(2)})$$
$$\mathbf{c}^{(1)} = \mathbf{V}^{-1}\,\mathbf{A}\,\boldsymbol{\lambda}^{(1)},\ \mathbf{c}^{(2)} = \mathbf{V}^{-1}\,\mathbf{A}\,\boldsymbol{\lambda}^{(2)}$$
$$(\mathbf{c}^{(1)} - \mathbf{c}^{(2)})'\,\mathbf{V}^{-1}\,(\mathbf{c}^{(1)} - \mathbf{c}^{(2)}) = (\boldsymbol{\lambda}^{(1)} - \boldsymbol{\lambda}^{(2)})'\,\mathbf{A}'\,\mathbf{V}^{-1}\,\mathbf{A}\,(\boldsymbol{\lambda}^{(1)} - \boldsymbol{\lambda}^{(2)}) = \mathbf{O}$$
$$\therefore\ \mathbf{c}^{(1)} = \mathbf{c}^{(2)}$$

A particular solution of (1.8.2) is $\boldsymbol{\lambda} = \begin{bmatrix} (\mathbf{A}'_1\,\mathbf{V}^{-1}\,\mathbf{A}_1)^{-1}\,\mathbf{b}_1 \\ \mathbf{O} \end{bmatrix}$

Hence from what we have proved above

$$\mathbf{c} = \mathbf{V}^{-1}\ \mathbf{A}_1\,(\mathbf{A}'_1\,\mathbf{V}^{-1}\,\mathbf{A}_1)^{-1}\,\mathbf{b}_1 \tag{1.8.3}$$

For all $\mathbf{c}$ such that $\mathbf{A}'\,\mathbf{c} = \mathbf{b}$, we have

$$\begin{aligned} \mathbf{c}'\,\mathbf{V}\,\mathbf{c} &= \mathbf{c}'\,[\,\mathbf{V} - \mathbf{A}_1\,(\mathbf{A}'_1\,\mathbf{V}^{-1}\,\mathbf{A}_1)^{-1}\,\mathbf{A}'_1]\,\mathbf{c} + \mathbf{b}'_1\,(\mathbf{A}'_1\,\mathbf{V}^{-1}\,\mathbf{A}_1)^{-1}\,\mathbf{b}_1 \\ &= \mathbf{c}'\mathbf{H}'\,[\,\mathbf{I} - \mathbf{H}'^{-1}\,\mathbf{A}_1(\mathbf{A}'_1\,\mathbf{V}^{-1}\,\mathbf{A}_1)^{-1}\,\mathbf{A}'_1\,\mathbf{H}^{-1}]\,\mathbf{H}\,\mathbf{c} + \mathbf{b}'_1(\mathbf{A}'_1\,\mathbf{V}^{-1}\,\mathbf{A}_1)^{-1}\,\mathbf{b}_1 \\ &= \mathbf{T}'\,\mathbf{T} + \mathbf{b}'_1\,(\mathbf{A}'_1\,\mathbf{V}^{-1}\,\mathbf{A}_1)^{-1}\,\mathbf{b}' \end{aligned}$$

Where $\mathbf{T}$ is the $n \times 1$ matrix

$$[\mathbf{I} - \mathbf{H}'^{-1}\,\mathbf{A}_1\,(\mathbf{A}'_1\,\mathbf{V}^{-1}\,\mathbf{A}_1)^{-1}\,\mathbf{A}'_1\,\mathbf{H}^{-1}]\,\mathbf{H}\,\mathbf{c}$$
$$\therefore\ \mathbf{c}'\,\mathbf{V}\,\mathbf{c} \geqslant \mathbf{b}'_1\,(\mathbf{A}'_1\,\mathbf{V}^{-1}\,\mathbf{A}_1)^{-1}\,\mathbf{b}_1$$

and the minimum is actually assumed for (1.8.3).

It is easy to see that the best linear unbiased estimate of $\mathbf{b}'\,\boldsymbol{\theta}$ is

$$\mathbf{b}'\,\hat{\boldsymbol{\theta}} \tag{1.8.4}$$

where $\hat{\boldsymbol{\theta}}$ is the value of $\boldsymbol{\theta}$ which will minimise

$$(\mathbf{y} - \mathbf{A}\,\boldsymbol{\theta})'\,\mathbf{V}^{-1}\,(\mathbf{y} - \mathbf{A}\,\boldsymbol{\theta}) \tag{1.8.5}$$

Differentiating (1.8.5) with respect to $\boldsymbol{\theta}$ and equating the derivative to $\mathbf{O}$, we get

$$\mathbf{A}'\mathbf{V}^{-1}\mathbf{y} = \mathbf{A}'\mathbf{V}^{-1}\mathbf{A}\hat{\boldsymbol{\theta}} \tag{1.8.6}$$

Now from (1.8.1), (1.8.2) and (1.8.6)

$$\mathbf{c}'\mathbf{y} = \boldsymbol{\lambda}'\mathbf{A}'\mathbf{V}^{-1}\mathbf{y} = \boldsymbol{\lambda}'\mathbf{A}'\mathbf{V}^{-1}\mathbf{A}\hat{\boldsymbol{\theta}} = \mathbf{b}'\hat{\boldsymbol{\theta}} \tag{1.8.7}$$

Also from (1.8.1), (1.8.2) and (1.8.3)

$$\begin{aligned} V(\mathbf{c}'\mathbf{y}) &= \mathbf{c}'\mathbf{V}\mathbf{c} \\ &= \boldsymbol{\lambda}'\mathbf{A}'\mathbf{V}^{-1}\mathbf{A}\boldsymbol{\lambda} = \boldsymbol{\lambda}'\mathbf{b} \qquad (1.8.8) \\ &= \mathbf{b}'_1(\mathbf{A}'_1\mathbf{V}^{-1}\mathbf{A}_1)^{-1}\mathbf{b}_1 \qquad (1.8.9) \end{aligned}$$

A linear function $\mathbf{l}'\mathbf{y}$ will be said to belong to error when $E(\mathbf{l}'\mathbf{y}) = \mathbf{O}$ irrespective of the values of the parameters. Thus if $\mathbf{l}'\mathbf{y}$ belongs to error, $\mathbf{A}'\mathbf{l} = \mathbf{O}$. There are $n - r$ independent linear functions of the $y$'s belong to error. The best unbiased estimates are $\mathbf{c}'\mathbf{y}$ where $\mathbf{c} = \mathbf{V}^{-1}\mathbf{A}\boldsymbol{\lambda}$. Covariance between $\mathbf{c}'\mathbf{y}$ and $\mathbf{l}'\mathbf{y}$ is $\mathbf{l}'\mathbf{V}\mathbf{c} = \mathbf{l}'\mathbf{A}\mathbf{y} = \mathbf{O}$. Thus the best unbiased linear estimates are uncorrelated with linear functions belonging to error.

In (1.8.6), no matter which solutions $\hat{\boldsymbol{\theta}}$ we take, $\mathbf{H}\mathbf{A}\hat{\boldsymbol{\theta}}$ remains the same. Hence

$$\hat{\boldsymbol{\theta}}'\mathbf{A}'\mathbf{V}^{-1}\hat{\boldsymbol{\theta}} \tag{1.8.10}$$

remains the same. But since

$$(\mathbf{y} - \mathbf{A}\hat{\boldsymbol{\theta}})'\mathbf{V}^{-1}(\mathbf{y} - \mathbf{A}\hat{\boldsymbol{\theta}}) = \mathbf{y}'\mathbf{V}^{-1}\mathbf{y} - \hat{\boldsymbol{\theta}}'\mathbf{A}'\mathbf{V}^{-1}\hat{\boldsymbol{\theta}} \tag{1.8.11}$$

it also remains the same no matter which solution $\hat{\boldsymbol{\theta}}$ of (1.8.6) we take. A particular solution of (1.8.6) is

$$\begin{bmatrix} (\mathbf{A}'_1\mathbf{V}^{-1}\mathbf{A}_1)^{-1}\mathbf{A}'_1\mathbf{V}^{-1}\mathbf{y} \\ \mathbf{O} \end{bmatrix} \tag{1.8.12}$$

Using (1.8.12)

$$\begin{aligned} &(\mathbf{y} - \mathbf{A}\boldsymbol{\theta})'\mathbf{V}^{-1}(\mathbf{y} - \mathbf{A}\hat{\boldsymbol{\theta}}) \\ &= (\mathbf{y} - \mathbf{A}\boldsymbol{\theta})'\mathbf{H}^{-1}[\mathbf{I} - \mathbf{H}\mathbf{A}_1(\mathbf{A}'_1\mathbf{A}_1)^{-1}\mathbf{A}'_1\mathbf{H}']\mathbf{H}^{-1}(\mathbf{y} - \mathbf{A}\boldsymbol{\theta}) \qquad (1.8.13) \end{aligned}$$

$\mathbf{H}'^{-1}(\mathbf{y} - \mathbf{A}\boldsymbol{\theta})$ are independent normal variates with zero means and unit variance and

$$\mathbf{I} - \mathbf{H}\mathbf{A}_1(\mathbf{A}'_1\mathbf{A}_1)^{-1}\mathbf{A}'_1\mathbf{H}'$$

is a symmetric indempotent matrix of rank $n - r$. Hence

$$E(\mathbf{y} - \mathbf{A}\boldsymbol{\theta})'\mathbf{V}^{-1}(\mathbf{y} - \mathbf{A}\boldsymbol{\theta}) = n - r \tag{1.8.14}$$

## Exercises to Chapter 1

1. Given three independent stochastic variates $y_1, y_2, y_3$, having a common variance $\sigma^2$ such that $E(y_1) = \theta_1 + \theta_2$, $E(y_2) = \theta_1 + \theta_3$ and $E(y_3) = \theta_1 + \theta_2$, show that $1_1\theta_1 + 1_2\theta_2 + 1_3\theta_3$ is estimable if and only if $1_1 = 1_2 + 1_3$. Find an unbiased estimate of $\sigma^2$.

2. Let $y_i$, $i = 1, 2, \ldots, 8$ be independent stochostic variates having a common variance $\sigma^2$ and expectations given by $E(y_1) = \theta_1 + \theta_5$, $E(y_2) = \theta_2 + \theta_5$, $E(y_3) = \theta_3 + \theta_6$, $E(y_4) = \theta_4 + \theta_6$, $E(y_5) = \theta_1 + \theta_7$, $E(y_6) = \theta_3 + \theta_7$, $E(y_7) = \theta_2 + \theta_8$ and $E(y)_8 = \theta_4 + \theta_8$. Show that $\theta_1 - \theta_2$ is estimable and find at least four unbiased estimates of $\theta_1 - \theta_2$. Find also the best unbiased linear estimate and its variance. Establish further that $\theta_1 + \theta_2$ is not estimable. Obtain an unbiased estimate of $\sigma^2$.

3. Assume that $y_i$, $i = 1, 2, \ldots, 6$ are independent and have a common variance $\sigma^2$. Given $E(y_1) = E(y_2) = \theta_1$, $E(y_3) = E(y_4) = \theta_2$, $E(y_5) = E(y_6) = \theta_1 + \theta_2$, obtain the best unbiased linear estimates of $\theta_1$ and $\theta_2$. Find four orthogonal linear functions each belonging to error. Hence represent $\sum_{i=1}^{6} y_i^2$ as a sum of squares of six mutually orthogonal linear functions of the $y$'s.

4. It is known that the period of oscillation $t$ of a pendulum is proportional to the square root of its length $l$. In order to obtain the constant of proportionality, an experimenter chooses lengths $l_1, l_2, \ldots, l_k$ of the pendulum and makes $n_1, n_2 \ldots, n_k$ observations respectively corresponding to these lengths. Assuming that the observations are equally reliable, obtain the best unbiased linear estimate of the constant of proportionality and an estimate of its variance.

5. Given $0 \leqslant \alpha_i < \pi$ and $\alpha_i \neq \alpha_k$ for $i \neq k$ and $y_i$ are independent variates having variance $\sigma^2$ and $E(y_i) = \theta_1 \cos \alpha_i + \theta_2 \sin \alpha_i$, $i = 1, 2, \ldots, n$ obtain the best unbiased linear estimates of $\theta_1$ and $\theta_2$, their variances and the covariance between them. Show that $\sum\sum_{j>i} (y_i \cos \alpha_j - y_j \cos \alpha_i) \Big/ \frac{n(n-1)}{2}$ is an unbiased estimate of $\theta_2$ and comparing its variance with the variance of the best unbiased linear estimate of $\theta_2$ obtain the trigonometric inequality

$$\left[ \sum_{i=1}^{n} \cos^2 \alpha_i \Big/ \sum\sum_{j>i} \sin(\alpha_i - \alpha_j) \right] \leqslant \left[ \frac{2}{n(n-1)} \right]^2 \sum_{i=1}^{n} \left\{ \sum{}' \frac{\cos \alpha_j}{\sin(\alpha_i - \alpha_j)} \right\}^2$$

where the summation $\Sigma'$ extends over all values of $j$ except $i$.

6. $E(y_{ij}) = \mu_i$, $i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, n_i$. Making usual assumptions about $y_{ij}$ obtain separate tests for

$$(i)\ \mu_1 = \mu_2 = \ldots = \mu_k$$
$$\text{and } (ii)\ \mu_1 = \mu_2 = \ldots = \mu_k = 0.$$

7. $y$'s are independent normal variates having a common variance $\sigma^2$.

(*i*) Given $E(y_i) = \mu$, $i = 1, 2, \ldots, n$, obtain a test for $\mu = 0$.

(*ii*) Given $E(y_i) = \mu_1$, $i = 1, 2, \ldots, n_1$ and $E(y_i) = \mu_2$, $i = n_1 + 1, \ldots, n_1 + n_2$, obtain a test for $\mu_1 = \mu_2$.

8. $E(y_{ij}) = \mu + \alpha_i + \tau_j$ $i = 1, 2, \ldots, b$; $j = 1, 2, \ldots, v$. Making usual assumptions about $y_{ij}$'s

(*i*) Show that $l_1 \tau_1 + l_2 \tau_2 + \ldots + l_v \tau_v$ is estimable if and only if $l_1 + l_2 + \ldots + l_v = 0$

(*ii*) Obtain a test for $\tau_1 = \tau_2 = \ldots = \tau_v = 0$.

9. $E(y_{ij}) = \mu + \alpha_i + \tau_j + \beta\,(x_{ij} - \bar{x})$, $i = 1, 2, \ldots, p$; $j = 1, 2, \ldots, q$; $x_{ij}$'s are known constants, $\bar{x} = \sum_{i=1}^{p} \sum_{j=1}^{q} x_{ij} / p\,q$ and $y_{ij}$'s are independent normal having a common variance $\sigma^2$. Obtain the best unbiased linear estimate of $\beta$ and a test for $\beta = 0$. Show that $l_1 \tau_1 + l_2 \tau_2 + \ldots + l_v \tau_v$ is estimable if and only if $l_1 + l_2 + \ldots + l_v = 0$. Obtain the best unbiased linear estimate of $\tau_1 = \tau_2$ and its variance. Indicate a test for the hypothesis $\tau_1 = \tau_2 = \ldots = \tau_v = 0$.

10. Given known constants $x_{ij}$, ($i=1, 2, \ldots, k$; $j = 1, 2, \ldots, n_i$), $\bar{x}_i = \sum_{j=1}^{n_i} x_{ij} / n_i$, $\bar{x} = \sum_{i=1}^{k} n_i \bar{x}_i / (n_1 + n_2 + \ldots + n_k)$, $y_{ji}$'s to be independent normal variates having a common variance $\sigma^2$ and $E(y_{ij}) = \alpha_i + \beta(x_{ij} - \bar{x}_i)$,

(*a*) Obtain a test for the hypothesis $\beta_1 = \beta_2, = \ldots = \beta_k$

(*b*) Assuming $\beta_1 = \beta_2 = \ldots = \beta_k = \beta$ (say), test

(*i*) $\alpha_i = \alpha + \beta_m (\bar{x}_i - \bar{x})$

(*ii*) $\beta = \beta_m$

11. $E(y_i) = \alpha + \beta(x_i - \bar{x})$, ($i = 1, 2, \ldots, n$), $x$'s being constants, $\bar{x} = (x_1 + x_2 + \ldots + x_n)/n$ and $y$'s are independent normal having a common variance $\sigma^2$.

(*i*) Obtain best unbiased linear estimates for $\alpha$ and $\beta$.

(*ii*) Derive separate tests for $\alpha = 0$ and $\beta = 0$.

12. $y_1, y_2, \ldots, y_n$ are independent normal having a common variance $\sigma^2$ and $E(y_i) = \alpha + \beta_1(x_{1i} - \bar{x}_1) + \beta_2(x_{2i} - \bar{x}_2) + \ldots + \beta_p(x_{pi} - \bar{x}_p)$, $x_{ii}$'s being known constants, $\bar{x}_t = (x_{t1} + x_{t2} + \ldots + x_{tn}) / n$, $t = 1, 2, \ldots, p$ obtain best unbiased linear estimates of $\alpha, \beta_1, \beta_2, \ldots, \beta_p$ and find their dispersion matrix. Obtain tests for $(i)$ $\beta_1 = \beta_2$ and $(ii)$ $\beta_1 = \beta_2 = \ldots = \beta_p$.

13. $y_1, y_2, \ldots, y_n$ are independent normal having a common variance $\sigma^2$ and $E(y_i) = \alpha_0 X_0(x_i) + \alpha_1 X_1(x_i) + \ldots + \alpha_p X_p(x_i)$, $p < n$ and $X_0 = 1$, $X_j(x)$ for $j = 0, 1, 2, \ldots, p$ are orthogonal functions so that for $j \neq k$ $\sum_{i=1}^{n} X_j(x_i) X_k(x_i) = 0$, $x_1, x_2, \ldots, x_n$ being a set of constants no two of which are equal. Obtain best unbiased linear estimates of $\alpha_i$, $i = 0, 1, 2, \ldots, p$, and their variances. Derive a test for $\alpha_1 = 0$.

14. Let $y_1, y_2, \ldots, y_n$ be a random sample of size $n$ drawn without replacement from a finite population consisting of the elements $Y_1, Y_2 \ldots, Y_N$. Let $\bar{Y} = (Y_1 + Y_2 + \ldots + Y_N) / N$, $\sigma^2 = (Y_1^2 + Y_2^2 + \ldots + Y_N^2 - N\bar{Y}^2) / N$. Show that $(i)$ $E(y_i) = \bar{Y}$, $i = 1, 2, \ldots, n$ $(ii)$ Dispersion matrix $\mathrm{V}$ of $\mathrm{y} = \{y_1, y_2, \ldots, y_n\}$ is $[\sigma^2 / (N - 1)]\,(N\,\mathrm{I}_n - \mathrm{E}_{nn})$ and $\mathrm{V}^{-1} = \dfrac{N-1}{\sigma^2}\left[\mathrm{I}_n + \dfrac{1}{N-n}\mathrm{E}_{nn}\right]$. Find the best unbiased linear estimate of $\bar{Y}$ and an estimate of its variance.

15. $y_1, y_2, \ldots y_n$ are stochastic variates having dispersion matrix $\mathrm{V} = \mathrm{diag}\,(1/w_1, 1/w_2, \ldots, 1/w_n)$ and $E(y_i) = \alpha + \beta x_i$, $(i = 1, 2, \ldots, n)$ the $x$'s being constants. Find the best unbiased linear estimates of $\alpha$ and $\beta$, their variances and the covariance between them. Work out the particular case $x_i = i$, $w_i = 1/i$, $i = 1, 2, \ldots, n$.

16. Let a finite population consisting of $N = N_1 + N_2 + \ldots + N_k$ elements be divided into $k$ strata, the $i$ th stratum consisting of $N_i$ elements $i = 1, 2, \ldots, k$. Let $Y_{ij}$, $(i = 1, 2, \ldots, k;\ j = 1, 2, \ldots, N_i)$ denote the value of a characteristic of the $j$ th element of $i$ th stratum.

Let $\mu_i = \sum_{j=1}^{N_i} Y_{ij}/N_i$, $\sigma_i^2 = \sum_{j=1}^{N_i} (Y_{ij} - \bar{Y}_i)^2/N_i$, $\bar{Y} = \sum_{i=1}^{k} N_i \bar{Y}_i / \sum_{i=1}^{k} N_i$.

Let $\mathrm{y} = \left\{ y_{11}, y_{12}, \ldots, y_{1n_1}, y_{21}, y_{22}, \ldots, y_{2n_2}, \ldots, y_{k1}, y_{k2}, \ldots, y_{kn_k} \right\}$ be a random sample from this population, the sampling being without replacement and there being $n_i$ elements from the $i$ th stratum and let $\mu = \{\mu_1, \mu_2, \ldots, \mu_k\}$ Show that

$(i)$ $E(\mathrm{y}) = \mathrm{diag}\,(\mathrm{E}_{n_1 1}, \mathrm{E}_{n_2 1}, \ldots, \mathrm{E}_{n_k 1})\,\mu$

$(ii)$ $\mathrm{V}(\mathrm{y}) = \mathrm{diag}\,[N_1 \mathrm{I}_{n_1} - \mathrm{E}_{n_1 n_1})\sigma_1^2/(N_1 - 1), \ldots, (N_k \mathrm{I}_{n_k} - \mathrm{E}_{n_k n_k})\,\sigma_k^2/(N_k - 1)\,]$

$$(iii)\ \mathbf{V}^{-1}(\mathbf{y}) = \text{diag}\left[\frac{N_1-1}{N\sigma_1^2}\left\{\mathbf{I}_{n_1}+\frac{1}{N_1-n_1}\mathbf{E}_{n_1 n_1}\right\}, \ldots, \frac{N_k-1}{N\sigma_k^2}\left\{\mathbf{I}_{n_k}+\frac{1}{N_k-n_k}\mathbf{E}_{n_k n}\right\}\right]$$

Find the best unbiased linear estimates of

$(N_1\mu_1 + N_2\mu_2 + \ldots + N_k\mu_k) / \sum_{i=1}^{k} N_i$ and obtain an unbiased estimate of its variance.

17. We are interested in estimating the population mean of a character $y$. For increasing the precision of the estimate, we utilise an auxillary variate $x$ which is highly correlated with $y$. Suppose that the finite population of $y$ is divided into $k$ classes according to the value $x_i$ of $x$, $i = 1, 2, \ldots k$, there being $N_i$ values of $y$ corresponding to $x = x_i$, $N = N_1 + N_2 + \ldots + N_k$. A random sample of size $n$ is drawn, from the population, the number of units of $y$ having the value $x_i$ being $n_i$. Assume that (i) $E(y|x) = \alpha + \beta x$ (ii) $\text{Var}\,(y|x = x_i) = \gamma x_i$, (iii) the sampling within a class is carried out without replacement. Find the best unbiased linear estimate of $N(\alpha + \beta\bar{x})$, $\bar{x} = \sum_{i+1}^{k} N_i x_i | N$. Obtain the variance of this estimate and also an unbiased estimate of $\gamma$.

18. A finite population is divided into $M$ clusters, each cluster containing $N$ elements. Let a sample of $m$ clusters be chosen at random and from each selected cluster let $n$ units be chosen at random without replacement. Let $y_{ij}$ denote the value of a characteristic for the $j$ th individual chosen in the $i$th selected cluster $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$. Denoting by $\sigma^2$, the square of the population standard deviation, $\rho$, the population intra class correlation coefficient and by $\mathbf{y} = \{y_{11}, \ldots, y_{1n}, \ldots y_{m1}, y_{m2}, \ldots, y_{mn}\}$, the observed sample, show that

$$\mathbf{V}(\mathbf{y}) = \begin{pmatrix} \mathbf{V}_1\,\mathbf{V}_2\,\mathbf{V}_2 \cdots \mathbf{V}_2 \\ \mathbf{V}_2\,\mathbf{V}_1\,\mathbf{V}_2 \cdots \mathbf{V}_2 \\ \cdots\cdots\cdots\cdots \\ \mathbf{V}_2\,\mathbf{V}_2\,\mathbf{V}_2 \cdots \mathbf{V}_1 \end{pmatrix}, \mathbf{V}^{-1}(\mathbf{y}) = \begin{pmatrix} \mathbf{A}\ \mathbf{B}\ \mathbf{B} & \mathbf{B} \\ \mathbf{B}\ \mathbf{A}\ \mathbf{B} & \mathbf{B} \\ \cdots\cdots\cdots & \\ \mathbf{B}\ \mathbf{B}\ \mathbf{B} & \mathbf{A} \end{pmatrix}$$

where

$\mathbf{V}_1 = a\mathbf{I}_n + b\mathbf{E}_{nn}$, $\mathbf{V}_2 = c\mathbf{F}_{nn}$,

$$\mathbf{A} = (1/a)\,\mathbf{I}_n = \frac{\mathbf{E}_{nn}}{n}\left[\frac{1}{a} - \frac{a + nb + nc\,(m-2)}{(a + nb - nc)\,(a + nb - nc + ncm)}\right]$$

$\mathbf{B} = -\,b\mathbf{E}_{nn}/\,(a + nb - nc)\ (a + nb - nc + ncm)$, $a = \sigma^2\,(1-\rho)$, $b = \rho\,\sigma^2$ and $c = -\,\sigma^2\,[1 + \overline{n-1}\,\rho]\,/\,N\,(M-1)$.

Obtain the best unbiased linear estimate of the population mean and obtain its variance. Derive also an unbiased estimate of this variance from the sample values.

19. Let $x_1, x_2 \ldots, x_n$ be an ordered sample from a rectangular population with centre $\mu$ and range $\lambda$. The standard deviation $\sigma$ is evidently $(\lambda^2/12)^{\frac{1}{2}}$. Show that

(i) $E(x_i) = \mu + \sigma\alpha_i$ where $\alpha_i = \sqrt{3}\,(2i - n - 1) / (n + 1)$,
(ii) $\mathrm{V} = \sigma^2 (w_{ij})$, $w_{ij} = 12\, i\, (n - j + 1) / (n+1)^2 (n + 2)$, $i \leqslant j$

$$(iii)\quad [12\,\sigma^2/(n+1)\;(n + 2)]\; \mathrm{V}^{-1} = \begin{bmatrix} 2 & -1 & 0 & 0 \ldots 0 & 0 \\ -1 & 2 & -1 & 0 \ldots 0 & 0 \\ 0 & -1 & 2 & -1 \ldots 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 \ldots -1 & 2 \end{bmatrix}$$

Find the best unbaised linear estimates of $\mu$ and $\sigma$ and show that the two estimates are uncorrelated.

20. Let $x_1\, x_2 \ldots x_n$ be a random sample of size $n$ from the rectangular population having centre $\mu$ and range $= 2\sqrt{3}\;\sigma$ and let $X_1, X_2, \ldots, X_k$ be the censored sample of size $k$ in which $X_k$ is the greatest observation. The $(n-k)$ censored observations are known to be greater than $X_k$. Let the $k$ observations be arranged in increasing order of magnitude, the $r$ th smallest value being $X_{r|k}$. Thus

$$X_{1|n} < X_{2|n} \ldots < X_{k|n}$$

Show that $\mathrm{V}^{-1}$, the inverse of the dispersion matrix of $X_{1|n}, X_{2|n}, X_{k|n}$ is

$$\frac{(n+1)\,(n+2)}{12\,\sigma^2} \begin{bmatrix} 2 & -1 & 0 \ldots 0 & 0 & 0 \\ -1 & 2 & -1 \ldots 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 \ldots -1 & 2 & -0 \\ 0 & 0 & 0 \ldots 0 & -1 & \dfrac{n-k+2}{n-k+1} \end{bmatrix}$$

Obtain the best unbiased linear estimate $\mu$ and $\sigma$ from the censored sample and find the variances of these estimates and the covariance between them.

21. Develop the theory of Linear Estimation when the parameters are subject to linear restrictions.

## CHAPTER II

# GENERAL STRUCTURE OF ANALYSIS OF DESIGNS

1. *General Theory of Analysis of Experimental Designs.* Let there be $v$ treatments and $b$ blocks having $k_1, k_2, \ldots, k_b$ plots respectively and let the $i$th treatment be replicated $r_i$ times ($i = 1, 2, \ldots, v$). Evidently

$$r_1 + r_2 + \ldots + r_v = k_1 + k_2 + \ldots + k_b = n \tag{2.1.1}$$

Let $n_{ij}$ denote the number of times the $i$th treatment occurs in the $j$th block ($i = 1, 2, \ldots, v$; $j = 1, 2, \ldots, b$). Then

$$\mathbf{N} = [n_{ij}] \tag{2.1.2}$$

will be called the incidence matrix of the design. Now

$$\mathbf{E}_{1v}\mathbf{N} = (k_1, k_2, \ldots, k_b) \tag{2.1.3}$$

$$\mathbf{N}\,\mathbf{E}_{b1} = \{r_1, r_2, \ldots, r_v\} \tag{2.1.4}$$

Let the total yields of the blocks be denoted, by $B_1, B_2, \ldots, B_b$ respectively and the total yields of the treatments by $T_1, T_2 \ldots, T_v$ respectively. Let

$$\mathbf{B} = \{B_1, B_2, \ldots, B_b\} \tag{2.1.5}$$

$$\mathbf{T} = \{T_1, T_2, \ldots, T_v\} \tag{2.1.6}$$

$$\mathbf{G} = \mathbf{E}_{1b}, \mathbf{B} = \mathbf{E}_{1v}\ \mathbf{T} \tag{2.1.7}$$

We shall assume that the yields of the plots are independently and normally distributed with a common variance $\sigma^2$ and expectations given by the sum of a general effect $\mu$, the effect of the block containing the plot and the effect of the treatment applied to the plot. Let the effect of the blocks and the effect of the treatments be denoted respectively by

$$\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, \ldots, \alpha_b\} \tag{2.1.8}$$

$$\boldsymbol{\tau} = \{\tau_1, \tau_2, \ldots, \tau_v\} \tag{2.1.9}$$

In this case the normal equations (1.3.2) are

$$\begin{bmatrix} \mathbf{G} \\ \mathbf{B} \\ \mathbf{T} \end{bmatrix} = \begin{bmatrix} n\mathbf{I}_1, \mathbf{E}_{1b}\operatorname{diag}(k_1, \ldots, k_b), \mathbf{E}_{1v}\operatorname{diag}(r_1, \ldots, r_v) \\ \operatorname{diag}(k_1, \ldots, k_b)\,\mathbf{E}_{b1}, \operatorname{diag}(k_1, \ldots, k_b), \mathbf{N}' \\ \operatorname{diag}(r_1, \ldots, r_v)\,\mathbf{E}_{v1}, \mathbf{N}, \operatorname{diag}(r_1, \ldots, r_v) \end{bmatrix} \begin{bmatrix} \hat{\mu} \\ \hat{\boldsymbol{\alpha}} \\ \hat{\boldsymbol{\tau}} \end{bmatrix} \tag{2.1.10}$$

The rank of the first matrix on the right hand side of (2.1.10) is the same as that of

$$\begin{bmatrix} \text{diag}(k_1, k_2, \dots, k_b), \mathbf{N}' \\ \mathbf{N}, \quad \text{diag}(r_1, \dots, r_v) \end{bmatrix} \tag{2.1.11}$$

and the rank of (2.1.11) is the same as the rank of either

$$\begin{bmatrix} \text{diag}(k_1, k_2, \dots, k_b) & \mathbf{N}' \\ \mathbf{O} & \mathbf{C} \end{bmatrix}. \tag{2.1.12}$$

or

$$\begin{bmatrix} D & \mathbf{N}' \\ \mathbf{O} & \text{diag}(r_1, r_2, \dots, r_v) \end{bmatrix} \tag{2.1.13}$$

where

$$\mathbf{C} = \text{diag}(r_1, r_2, \dots, r_v) - \mathbf{N}\text{diag}\left(\frac{1}{k_1}, \frac{1}{k_2}, \dots, \frac{1}{k_b}\right)\mathbf{N}' \tag{2.1.14}$$

$$\mathbf{D} = \text{diag}(k_1 k_2, \dots, k_b) - \mathbf{N}'\,\text{diag}\left(\frac{1}{r_1}, \frac{1}{r_2}, \dots, \frac{1}{r_v}\right)\mathbf{N} \tag{2.1.15}$$

(2.1.12) and (2.1.13) can be obtained by pre-multiplication and post-multiplication of (2.1.11) by the non-singular matrices

$$\begin{bmatrix} \mathbf{I}_b, & \mathbf{O} \\ -\mathbf{N}\,\text{diag}\left(\frac{1}{k_1}, \dots, \frac{1}{k_b}\right), & \mathbf{I}_v \end{bmatrix} \text{ and } \begin{bmatrix} \mathbf{I}_b, & \mathbf{O} \\ -\text{diag}\left(\frac{1}{r_1}, \dots, \frac{1}{r_v}\right)\mathbf{N}, & \mathbf{I}_v \end{bmatrix}$$

respectively. Hence we get the result

$$b + \text{rank } \mathbf{C} = v + \text{rank } \mathbf{D} \tag{2.1.16}$$

**C** and **D** are obviously symmetric matrices and

$$\mathbf{C}\,\mathbf{E}_{v1} = \mathbf{O},\ \mathbf{E}_{1v}\,\mathbf{C} = \mathbf{O},\ \mathbf{D}\,\mathbf{E}_{b1} = \mathbf{O},\ \mathbf{E}_{1b}\,\mathbf{D} = \mathbf{O}. \tag{2.1.17}$$

The rank of the first matrix on the right hand side of (2.1.10) with an additional column $\left(\frac{\mathbf{O}}{\mathbf{L}}\right)$ where $\mathbf{L} = \left\{l_1, l_2, \dots, l_v\right\}$ is the same as that of

$$\begin{bmatrix} \text{diag}(k_1, k_2, \dots, k_b), & \mathbf{N}', & \mathbf{O} \\ \mathbf{O}, & \mathbf{C}, & \mathbf{L} \end{bmatrix} \tag{2.1.18}$$

In order that (2.1.18) may have the same rank as (2.1.12), it is necessary that $l_1 + l_2 + \dots + l_v = 0$. Hence a necessary condition for the estimability of $\mathbf{L}'\tau$ is that $\mathbf{E}_{1v}\,\mathbf{L} = \mathbf{O}$. Such a function of the treatment effects will be called a treatment contrast. When rank $\mathbf{C} = v-1$, every treatment contrast

will be estimable. A similar consideration will show that a linear function of the block effects in order to be estimable must be a block contrast and when rank $\mathbf{C} = v-1$, by (2.1.16) rank $\mathbf{D} = b-1$ and every block contrast is estimable. Incidentally, it may observed that if rank $\mathbf{C} = v - t$, a set of $(t-1)$ independent treatment contrasts $\mathbf{L}'\boldsymbol{\tau}$ which are not estimable can be obtained from

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{E}_{1v} \end{bmatrix} \mathbf{L} = \mathbf{O} \tag{2.1.19}$$

A necessary and sufficient condition for every block contrast and treatment contrast to be estimable is that rank $\mathbf{C} = v - 1$. When this condition is satisfied, the design is said to be connected. Pre-multiplying (2.1.10) by

$$\left[ \mathbf{O}, -\mathbf{N} \operatorname{diag} \left( \frac{1}{k_1}, \ldots, \frac{1}{k_b} \right), \mathbf{I}_v \right]$$

we get

$$\mathbf{Q} = \mathbf{T} - \mathbf{N} \operatorname{diag} \left( \frac{1}{k_1}, \frac{1}{k_2}, \ldots, \frac{1}{k_b} \right) \mathbf{B} \tag{2.1.20}$$

$$= \mathbf{C}\,\hat{\boldsymbol{\tau}} \tag{2.1.21}$$

Evidently

$$E(\mathbf{Q}) = \mathbf{C}\boldsymbol{\tau},\ V(\mathbf{Q}) = \sigma^2 \mathbf{C},\ \mathbf{E}_{1v}\mathbf{Q} = \mathbf{O} \tag{2.1.22}$$

Similarly pre-multiplying $(2\cdot1\cdot10)$ by $\left[ \mathbf{O}, \mathbf{I}_b, -\mathbf{N} \operatorname{diag} \left( \frac{1}{r_1}, \ldots, \frac{1}{r_v} \right) \right]$

we get

$$\mathbf{P} = \mathbf{B} - \mathbf{N}' \operatorname{diag} \left( \frac{1}{r_1}, \frac{1}{r_2}, \ldots, \frac{1}{r_v} \right) \mathbf{T} \tag{2.1.23}$$

$$= \mathbf{D}\,\hat{\boldsymbol{\alpha}} \tag{2.1.24}$$

Obviously

$$\mathbf{E}(\mathbf{P}) = \mathbf{D}\boldsymbol{\alpha},\ \mathbf{V}(\mathbf{P}) = \sigma^2 \mathbf{D},\ \mathbf{E}_{1b}\mathbf{P} = \mathbf{O} \tag{2.1.25}$$

$\mathbf{Q}$ and $\mathbf{P}$ as defined in (2.1.20) and (2.1.23) are known respectively as adjusted treatment yields and adjusted block yields. We can also easily verify that in order that adjusted treatment and adjusted block yields may be mutually orthogonal, we must have

$$\mathbf{C} \operatorname{diag} \left( \frac{1}{r_1}, \frac{1}{r_2}, \ldots, \frac{1}{r_v} \right) \mathbf{N} = \mathbf{O} \tag{2.1.26}$$

$$\text{or } \mathbf{N} \operatorname{diag} \left( \frac{1}{k_1}, \frac{1}{k_2}, \ldots, \frac{1}{k_b} \right) \mathbf{D} = \mathbf{O} \tag{2.1.27}$$

Thus in order that the adjusted block yields may be orthogonal to the adjusted treatment yields, the design is either ( i ) not connected or (ii) the incidence matrix N is such that $n_{ij}/r_i$ is constant for all $i$. This also leads to the condition $n_{ij}/k_j$ = constant for all $j$. Hence consistent with conditions of the design no $n_{ij}$ can be zero in this case. Defining an incomplete block design as one in which at least one of the blocks does not contain all the treatments, we can assert that in a connected incomplete block design all the adjusted block yields can not be orthogonal to all the adjusted treatment yields.

In a large variety of designs, we have $k_1 = k_2 = \ldots = k_b = k$ ( say ) and each $n_{ij}$ is $o$ or 1. In this case from ( 2.1.14 ) we have

$$\mathbf{C} = \text{diag}\,(r_1\,,\, r_2\,, \ldots,\, r_v) - \frac{1}{k}\,\mathbf{N}\,\mathbf{N}' \tag{2.1.28}$$

Equation (2.1.21) now can be re-written as

$$Q_s = r_s \left[ 1 - \frac{1}{k} \right] \hat{\tau}_s - \frac{\lambda_{s1}}{k} \hat{\tau}_1 - \ldots - \frac{\lambda_{ss-1}}{k} \hat{\tau}_{s-1} - \frac{\lambda_{ss+1}}{k} \hat{\tau}_{s+1} - \ldots - \frac{\lambda_{sv}}{k} \hat{\tau}_v \quad s = 1,\, 2,\, \ldots\, v \tag{2.1.29}$$

Where $\lambda_{ss'}$ denotes the number of times the $s$ and $s'$ th treatments occur together in a block. A particular set of solutions of (2.1.10) is

$$\hat{\mu} = G/n,\ \boldsymbol{\alpha} = \text{diag}\left[\frac{1}{k_1}, \ldots, \frac{1}{k_b}\right] \mathbf{B} - \mathbf{E}_{b1}\,\hat{\mu} - \text{diag}\left[\frac{1}{k_1}, \ldots, \frac{1}{k_b}\right] \mathbf{N}'\hat{\boldsymbol{\tau}} \tag{2.1.30}$$

Hence the sum of squares due to regression when $\mu, \alpha_1, \alpha_2, \ldots, \alpha_b, \tau_1, \tau_2, \ldots, \tau_v$ are fitted is

$$\begin{aligned} &G^2/n + \hat{\boldsymbol{\alpha}}'\,\mathbf{B} + \hat{\boldsymbol{\tau}}'\,\mathbf{T} \\ &= \mathbf{B}'\ \text{diag}\left[\frac{1}{k_1}, \frac{1}{k_2}, \ldots, \frac{1}{k_b}\right] \mathbf{B} + \hat{\boldsymbol{\tau}}'\,\mathbf{Q} \end{aligned} \tag{2.1.31}$$

The associated degrees of freedom is $b + v - t$. When $\boldsymbol{\tau} = \mathbf{O}$, a particular set of solutions of the new normal equations are

$$\hat{\boldsymbol{\mu}}\,(\mathbf{O}) = G/n,\ \hat{\boldsymbol{\alpha}}\,(\mathbf{O}) = \text{diag}\left[\frac{1}{k_1}, \ldots, \frac{1}{k_b}\right] \mathbf{B} - \frac{G}{n}\,\hat{\boldsymbol{\mu}}(\mathbf{O}) \tag{2.1.32}$$

Hence the sum of squares due to regression when only $\mu, \alpha_1, \alpha_2, \ldots, \alpha_b$ are fitted is

$$\mathbf{B}'\ \text{diag}\left[\frac{1}{k_1}, \ldots, \frac{1}{k_b}\right] \mathbf{B} \tag{2.1.33}$$

The associated degrees of freedom is $b$. Hence if $\boldsymbol{\tau} = \mathbf{O}$, the difference between (2.1.31) and (2.1.33)

$$\hat{\tau}'\,\mathbf{Q} = \sum_{s=1}^{v} \hat{\tau}_s\, Q_s \qquad (2.1.34)$$

is distributed as $\chi^2\,\sigma^2$ with $v-t$ degrees of freedom. The analysis of variance for testing treatment and block effects may be given in the following summary table.

TABLE 1

*Analysis of Variance Table (Intrablock Analysis)*

| Source | DF | SS | SS | DF | Source |
|---|---|---|---|---|---|
| Blocks (ignoring treatments) | $b-1$ | $\sum_{j=1}^{b} \frac{B_j^2}{k_j} - \frac{G^2}{n}$ | † | $b-t$ | Blocks (Eliminating treatments) |
| Treatments (eliminating blocks) | $v-t$ | $\sum_{s=1}^{v} \hat{\tau}_s\, Q_s$ | $\sum_{i=1}^{v} \frac{T_i^2}{r_i} - \frac{G^2}{n}$ | $v-1$ | Treatments (ignoring blocks) |
| Error | $n-b-v+t$ | † | $\rightarrow$ | $n-b-v+t$ | Error |
| Total | $n-1$ | $\sum y^2 - \frac{G^2}{n}$ | $\rightarrow$ | $n-1$ | Total |

† Obtained by subtraction.

2. *Two-way of Elimination of Heterogeneity.* In some designs the position within the block is important as a source of variation and efficiency is gained by eliminating the positional effect. Consider a two-way design with $u$ rows and $u'$ columns. Let there be $v$ treatments, the $i$ th treatment being replicated $r_i$ times. Let $l_{ij}$ denote the number of times the $i$ th treatment occurs in the $j$ th row and $m_{ij'}$ denote the number of times the $i$ th treatment occurs in $j'$ th column. Let $\mathbf{L} = (l_{ij})$, $\mathbf{M} = (m_{ij'})$. Let $\mathbf{R} = \{R_1, R_2, \ldots\ldots, R_u\}$, $\mathbf{C} = \{C_1, C_2, \ldots\ldots, C_{u'}\}$, $\mathbf{T} = \{T_1, T_2, \ldots\ldots, T_v\}$ denote respectively the row totals, column total and treatment totals and $G = \mathbf{R}\,\mathbf{E}_{1u} = \mathbf{C}\,\mathbf{E}_{1u'} = \mathbf{T}\,\mathbf{E}_{1v}$ be the grand total. Let the expectation of the yield of the plot in the $j$ th row and

$j'$ th column having treatment $i$ (say) be $\mu + \alpha_j + \beta_j + \hat{\tau}_i$ where $\alpha_j$ = effect of $j$ th row, $\beta_{j'}$ = effect of $j'$ th column and $\tau_i$ = effect of $i$ th treatment. Let $\boldsymbol{\alpha} = \{ \alpha_1, \alpha_2, \ldots\ldots, \alpha_u \}$, $\boldsymbol{\beta} = \{ \beta_1, \beta_2, \ldots\ldots, \beta_{u'} \}$, $\boldsymbol{\tau} = \{ \tau_1, \tau_2, \ldots, \tau_v \}$

The normal equations are

$$\begin{bmatrix} \mathbf{G} \\ \mathbf{R} \\ \mathbf{C} \\ \mathbf{T} \end{bmatrix} = \begin{bmatrix} uu'\,\mathbf{I}_1, & u'\,\mathbf{E}_{1u}, & u\,\mathbf{E}_{1u'}, & \mathbf{E}_{1v}\,\text{diag}\,(r_1, r_2, \ldots, r_v) \\ u'\,\mathbf{E}_{u1}, & u'\,\mathbf{I}_u, & \mathbf{E}_{uu'}, & \mathbf{L}' \\ u\,\mathbf{E}_{u'1}, & \mathbf{E}_{u'u}, & u\,\mathbf{I}_{u'}, & \mathbf{M}' \\ \text{diag}\,(r_1, r_2, \ldots, r_v)\,\mathbf{E}_{v1}, & \mathbf{L}, & \mathbf{M}, & \text{diag}\,(r_1, r_2, \ldots, r_v) \end{bmatrix} \begin{bmatrix} \hat{\mu} \\ \hat{\boldsymbol{\alpha}} \\ \hat{\boldsymbol{\beta}} \\ \hat{\boldsymbol{\tau}} \end{bmatrix} \qquad \ldots\ldots(2.2.1)$$

The rank of the first matrix on the right hand side of $(2\cdot2\cdot1)$ is the same as the rank of

$$\begin{bmatrix} u'\,\mathbf{I}_u, & \mathbf{E}_{uu'} & \mathbf{L}' \\ \mathbf{E}_{u'u}, & u\,\mathbf{I}_{u'}, & \mathbf{M}' \\ \mathbf{L}, & \mathbf{M}, & \text{diag}\,(r_1, r_2 \ldots, r_v) \end{bmatrix} \qquad (2.2.2)$$

(2.2.2) has the rank $u + u' - 1 + \text{rank } F$ which is the rank of

$$\begin{bmatrix} \mathbf{I}_u, & \frac{1}{u'}\,\mathbf{E}_{uu'}, & \mathbf{L}' \\ \mathbf{O}, & \mathbf{I}_{u'} - \frac{1}{u'}\,\mathbf{E}_{u'u'}, & \mathbf{M}' - \frac{E_{u'v}}{uu'}\,\text{diag}\,(r_1, \ldots, r_v) \\ \mathbf{O}, & \mathbf{O}, & \mathbf{F} \end{bmatrix} \qquad (2.2.3)$$

where

$$\mathbf{F} = \text{diag}\,(r_1, r_2, \ldots, r_v) - \frac{1}{u'}\,\mathbf{LL}' - \frac{1}{u}\,\mathbf{MM}' + \frac{1}{uu'}\,\mathbf{L}\,\mathbf{E}_{uu}\,\mathbf{L}' \qquad (2.2.4)$$

The result is obvious for (2.2.3) can be obtained by pre-multiplying (2.2.2) by the non-singular matrix

$$\begin{bmatrix} \frac{1}{u'}\,\mathbf{I}_u, & \mathbf{O} & \mathbf{O} \\ \mathbf{O}, & \frac{1}{u}\,\mathbf{I}_{u'} & -\frac{1}{uu'}\,\mathbf{E}_{u'v} \\ -\frac{1}{u'}\,\mathbf{L} + \frac{\mathbf{LE}_{uu}}{uu'}, & -\frac{\mathbf{M}}{u}, & \mathbf{I}_v \end{bmatrix} \qquad (2.2.5)$$

Note that $\mathbf{F}$ as given in (2.2.4) is symmetric and

$$\mathbf{M}\,\mathbf{E}_{u'u'} = \mathbf{L}\,\mathbf{E}_{uu'},\ \mathbf{L}\,\mathbf{E}_{uu} = \mathbf{M}\,\mathbf{E}_{u'u},\ \mathbf{E}_{1v}\,\mathbf{F} = \mathbf{O},\ \mathbf{F}\,\mathbf{E}_{v1} = \mathbf{O} \qquad (2.2.6)$$

It is evident that in order that a linear function of the treatment effects be estimable, it must be a treatment contrast. When rank $\mathbf{F} = v - 1$, every treatment contrast is estimable. Pre-multiplying (2.2.1) by

$$\left[ \frac{\mathbf{L}\ \mathbf{E}_{u1}}{uu'},\ -\frac{\mathbf{L}}{u'},\ -\frac{\mathbf{M}}{u},\quad \mathbf{I}_v \right] \quad \text{we get}$$

$$\mathbf{Q} = \mathbf{F}\ \hat{\boldsymbol{\tau}} \tag{2.2.7}$$

where

$$\mathbf{Q} = \mathbf{T} - \frac{\mathbf{L}\ \mathbf{R}}{u'} - \frac{\mathbf{M}\ \mathbf{C}}{u} + \frac{\mathbf{L}\ \mathbf{E}_{u1}\ \mathbf{G}}{uu'} \tag{2.2.8}$$

$$\mathbf{E}_{1v}\ \mathbf{Q} = \mathbf{O},\ E(\mathbf{Q}) = \mathbf{F}\,\boldsymbol{\tau} \text{ and } \mathbf{V}(\mathbf{Q}) = \sigma^2\,\mathbf{F} \tag{2.2.9}$$

The row totals and column totals are orthogonal to $\mathbf{Q}$. Also every contrast of row totals is orthogonal to every contrast of column totals. Hence we have the following analysis of variance table.

TABLE 2

*Analysis of variance for two-way elimination of heterogeneity*

| Source | D F | S S |
|---|---|---|
| Rows | $u - 1$ | $\sum_{j=1}^{u} \frac{R_j^2}{u'} - \frac{G^2}{uu'}$ |
| Columns | $u' - 1$ | $\sum_{j'=1}^{u'} \frac{C_{j'}^2}{u} - \frac{G^2}{uu'}$ |
| Treatments (adjusted) | $v - t$ | $\sum_{s=1}^{v} \hat{\tau}_s\, Q_s$ |
| Error | $uu' - u - u' + 1 - v + t$ | † |
| Total | $uu' - 1$ | $\Sigma\, y^2 - G^2/uu'$ |

† Obtained from subtraction.

It is easy to verify that when

$$\mathbf{E}_{uu'} = \mathbf{L}'\ \text{diag}\left\{ \frac{1}{r_1},\ \ldots,\ \frac{1}{r_v} \right\}\ \mathbf{M} \tag{2.2.10}$$

the estimable row contrasts and the estimable column contrasts are mutually orthogonal. In this case, estimable row contrasts are also orthogonal to estimable treatment contrasts when

$$\left[ u' \mathbf{I}_u - \mathbf{L}' \ \text{diag} \ \left\{ \frac{1}{r_1}, \ldots, \frac{1}{r_v} \right\} \mathbf{L} \right] \mathbf{L}' = \mathbf{O} \qquad (2.2.11)$$

and the estimable column contrasts are orthogonal to the estimable treatment contrasts when

$$\left[ u \, \mathbf{I}_{u'} - \mathbf{M}' \ \text{diag} \ \left\{ \frac{1}{r_1}, \ldots, \frac{1}{r_v} \right\} \mathbf{M} \right] \mathbf{M}' = \mathbf{O} \qquad (2.2.12)$$

The estimable row and column contrasts are given respectively by

$$\left[ u' \ \mathbf{I}_u - \mathbf{L}' \ \text{diag} \ \left\{ \frac{1}{r_1}, \ldots, \frac{1}{r_v} \right\} \mathbf{L} \right] \boldsymbol{\alpha} \qquad (2.2.13)$$

and $$\left[ u \, \mathbf{I}_{u'} - \mathbf{M}' \ \text{diag} \ \left\{ \frac{1}{r_1}, \ldots, \frac{1}{r_v} \right\} \mathbf{M} \right] \boldsymbol{\beta} \qquad (2.2.14)$$

and their best unbiased linear estimates by

$$\mathbf{R} - \mathbf{L}' \ \text{diag} \ \left\{ \frac{1}{r_1}, \ldots, \frac{1}{r_v} \right\}' \mathbf{T} \qquad (2.2.15)$$

and $$\mathbf{C} - \mathbf{M}' \ \text{diag} \ \left\{ \frac{1}{r_1}, \ldots, \frac{1}{r_v} \right\} \mathbf{T} \qquad (2.2.16)$$

3. *Analysis with Recovery of Interblock Information.* The yield of the treatment in the $j$ th plot of the $i$ th block has in its expectation the term $\alpha_i$, the block effect, which was regarded as constant in the previous analysis. If $\alpha_i$ is itself a random variable which is normally distributed about zero as mean and standard deviation $\sigma_2$, the analysis becomes slightly changed. In what follows, we shall assume that there are $b$ blocks of $k$ plots each and $v$ treatments are assigned to these $bk$ plots, the $s$ th treatment being replicated $r_s$ times ( $s=1, 2, \ldots, v$ ). No treatment is assigned more than once to a block and the number of times $s$ and $s'$ treatments occur together in a block is $\lambda_{ss'}$. Let $n_{si}$ denote 1 if the $s$ th treatment occurs in the $i$ th block and $o$ therewise. Let $y_{ij}$ denote the yield of the $j$ th plot in the $i$ th block ( $i = 1, 2, \ldots, b; j = 1, 2, \ldots, k$ ). For fixed $\alpha_1, \alpha_2, \ldots, \alpha_b$ $y_{ij}'$ s are independently and normally distributed with a common variance $\sigma_1^2$ and expectations give by $\mu + \alpha_i + \tau_{(ij)}$ where $\mu$ represents a general effect, $\alpha_i$, the additional effect of $i$ th block and $\tau_{(ij)}$, the effect of the treatment applied to the $j$ th plot of the $i$ th block; $\alpha_l, \alpha, \ldots, \alpha_b$ are themselves normally distributed about zero as mean and standard derivation $\sigma_2$.

Making use of the following results on conditional expectation

$$E\,E\,(x \mid z) = E\,(x)$$

$$\text{Var}\; x = E\;\;\text{Var}\,(x \mid z) + \text{Var}\; E\,(x \mid z)$$

$$\text{Cov}\,(x, y) = E\;\text{Cov}\,(x, y \mid z) + \text{Cov}\,[E\,(x \mid z), E\,(y \mid z)$$

we get easily

$$\text{E}\,(y_{ij}) = \mu + \tau_{(ij)} \tag{2.3.1}$$

$$\text{V}\,(y_{ij}) = \sigma_1^2 + \sigma_2^2 \tag{2.3.2}$$

$$\text{Cov}\,(y_{ij'}\;\; y_{i'j'}) = \begin{cases} \sigma_2^2 \text{ if } i = i',\ j \neq j' \\ o \text{ otherwise} \end{cases} \tag{2.3.3}$$

Hence if

$$\mathbf{y} = (y_{11}, y_{12}, \ldots, y_{1k}, y_{21}, \ldots, y_{2k}, \ldots, y_{b1}, y_{b2}, \ldots, y_{bk})$$

$$\mathbf{V}^{-1} = \text{diag}\;\left\{\frac{1}{\sigma_1^2}\,\mathbf{I}_k - \frac{\sigma_2^2}{\sigma_1^2\,(\sigma_1^2 + k\sigma_2^2)}\,\mathbf{E}_{kk}, \ldots, \frac{1}{\sigma_1^2}\,\mathbf{I}_k - \frac{\sigma_2^2}{\sigma_1^2\,(\sigma_1^2 + k\sigma_2^2)}\,\mathbf{E}_{kk}\right\}$$

$$\mathbf{V} = \text{diag}\;(\sigma_1^2\,\mathbf{I}_k + \sigma_2^2\,\mathbf{E}_{kk}, \ldots, \sigma_1^2\,\mathbf{I}_k + \sigma_2^2\,\mathbf{E}_{kk}) \tag{2.3.4}$$

$$\mathbf{E}\,(\mathbf{y}) = (E_{bk1}\;\boldsymbol{\alpha}_1\;\boldsymbol{\alpha}_2 \ldots \boldsymbol{\alpha}_v)\;\begin{bmatrix}\boldsymbol{\mu}\\ \boldsymbol{\tau}\end{bmatrix} = \mathbf{A}\;\begin{bmatrix}\boldsymbol{\mu}\\ \mathbf{T}\end{bmatrix} \tag{2.3.5}$$

Where $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots, \tau_v)$, $\boldsymbol{\alpha}_s$ is an $bk \times 1$ matrix consisting of zeros and $r_s$ **1**'s

$$\boldsymbol{\alpha}'_s\,\boldsymbol{\alpha}_s = r_s,\; \boldsymbol{\alpha}'_s\,\boldsymbol{\alpha}_{s'} = o,\; s \neq s' \tag{2.3.6}$$

$$\mathbf{A}'\,\mathbf{V}^{-1} = \begin{bmatrix} w'\;\mathbf{E}_{1bk} \\ w\;\boldsymbol{\alpha}'_1 - \dfrac{w - w'}{k}\;(n_{11}\;\mathbf{E}_{1k}, \ldots, n_{1b}\;\mathbf{E}_{1k}) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ w\;\boldsymbol{\alpha}'_v - \dfrac{w - w'}{k}\;(n_{v1}\;\mathbf{E}_{1k}, \ldots, n_{vb}\;\mathbf{E}_{1k}) \end{bmatrix} \tag{2.3.7}$$

$$\mathbf{A}'\,\mathbf{V}^{-1}\,\mathbf{y} = \begin{bmatrix} w'\,G \\ w\,Q_1 + \dfrac{w'}{k}\,B'_1 \\ \cdots\cdots\cdots\cdots \\ w\,Q_v + \dfrac{w'}{k}\,B'_v \end{bmatrix} \tag{2.3.8}$$

$$\mathbf{A}'\mathbf{V}^{-1}\mathbf{A}=\begin{bmatrix} bkw' & r_1 w' & r_2 w' & \dots & r_v w' \\ r_1 w', & r_1\left[w-\dfrac{w-w'}{k}\right], & -\dfrac{\lambda_{12}(w-w')}{k} & \dots & -\lambda_{1v}\dfrac{(w-w')}{k} \\ r_2 w', & -\dfrac{\lambda_{12}(w-w')}{k}, & r_2\left[w-\dfrac{w-w'}{k}\right] & \dots & -\lambda_{2v}\dfrac{w-w'}{k} \\ \dots & \dots & \dots & \dots & \dots \\ r_v w', & -\dfrac{\lambda_{1v}(w-w')}{k} & \dots & \dots & r_v\left[w-\dfrac{w-w'}{k}\right] \end{bmatrix} \tag{2.3.9}$$

where $B'_s=\sum_{i=1}^{b} n_{si}\ B_i/k$, $B_i$ and $Q_s$ having the same meanings as in (2.1.5) and (2.1.20) $w=1/\sigma_1^2$ and $w'=1/(\sigma_1^2+k\sigma_2^2)$. Since

$$\mathbf{A}'=\begin{bmatrix} \mathbf{E}_{1bk} \\ \boldsymbol{\alpha}'_1 \\ \dots \\ \boldsymbol{\alpha}'_v \end{bmatrix}, \text{ in order that rank } \mathbf{A}' = \text{rank}\begin{bmatrix} \mathbf{E}_{1bk} & 0 \\ \boldsymbol{\alpha}'_1 & l_1 \\ \dots & \dots \\ \boldsymbol{\alpha}'_v & l_v \end{bmatrix}$$

$l_1+l_2+\dots+l_v=0$. Conversely, if $l_1+l_2+\dots+l_v=0$

$$\text{rank}\begin{bmatrix} \mathbf{E}_{1bk} & 0 \\ \boldsymbol{\alpha}'_1 & l_1 \\ \boldsymbol{\alpha}'_2 & l_2 \\ \dots & \dots \\ \boldsymbol{\alpha}'_v & l_v \end{bmatrix}=\text{rank}\begin{bmatrix} \mathbf{0} & 0 \\ \boldsymbol{\alpha}'_1 & l_1 \\ \boldsymbol{\alpha}'_2 & l_2 \\ \dots & \dots \\ \boldsymbol{\alpha}'_v & l_v \end{bmatrix}=v=\text{rank}\begin{bmatrix} \mathbf{E}_{1bk} \\ \boldsymbol{\alpha}'_1 \\ \boldsymbol{\alpha}'_2 \\ \dots \\ \boldsymbol{\alpha}'_v \end{bmatrix}$$

Hence every treatment contrast is estimable. Now

$$\begin{bmatrix} r_1\left[w-\dfrac{w-w'}{k}\right], & -\dfrac{\lambda_{12}(w-w')}{k} & \dots & -\lambda_{1v}\left[\dfrac{w-w'}{k}\right] \\ -\lambda_{12}\left[\dfrac{w-w'}{k}\right], & r_2\left[w-\dfrac{w-w'}{k}\right] & \dots & -\lambda_{2v}\left[\dfrac{w-w'}{k}\right] \\ \dots & \dots & \dots & \dots \\ -\lambda_{1v}\left[\dfrac{w-w'}{k}\right], & -\lambda_{2v}\left[\dfrac{w-w'}{k}\right] & \dots & r_v\left[w-\dfrac{w-w'}{k}\right] \end{bmatrix}$$

is

$$w\,\mathbf{C}+\frac{w'}{k}\,\mathbf{NN}' \tag{2.3.10}$$

and is nothing but $\begin{bmatrix} \boldsymbol{\alpha}'_1 \\ \boldsymbol{\alpha}'_2 \\ \cdots \\ \boldsymbol{\alpha}'_v \end{bmatrix} \mathbf{V}^{-1} \begin{bmatrix} \boldsymbol{\alpha}_1 \boldsymbol{\alpha}_2 \ldots \boldsymbol{\alpha}_v \end{bmatrix}$, the rank

of which is $v$. Hence a particular solution of the normal equations

$$\begin{bmatrix} G\,w' \\ w\mathbf{Q} + \frac{w'}{k}\mathbf{NB} \end{bmatrix} = \begin{bmatrix} bkw' \,,\, w'\,\mathbf{E}_{1v}\,\mathrm{diag}\,(r_1, r_2, \ldots, r_v) \\ w'\,\mathrm{diag}\,(r_1, \ldots, r_v)\,\mathbf{E}_{v1},\, w\mathbf{C} + \frac{w'}{k}\mathbf{NN}' \end{bmatrix} \begin{bmatrix} \hat{\mu} \\ \hat{\boldsymbol{\tau}} \end{bmatrix} \qquad (2.2.11)$$

$$\hat{\mu} = o,\ \hat{\boldsymbol{\tau}} = \left[ w\mathbf{C} + \frac{w'}{k}\mathbf{NN}' \right]^{-1} \left[ w\,\mathbf{Q} + \frac{w'}{k}\mathbf{NB} \right] \qquad (2.3.12)$$

The best unbiased linear estimate of a treatment contrast $\mathbf{L}'\boldsymbol{\tau}$ is

$$\mathbf{L}' \left[ w\mathbf{C} + \frac{w'}{k}\mathbf{NN}' \right]^{-1} \left[ w\mathbf{Q} + \frac{w'}{k}\mathbf{NB} \right] \qquad (2.3.13)$$

and the variance of this estimate is

$$\mathbf{L}' \left[ w\mathbf{C} + \frac{w'}{k}\mathbf{NN}' \right]^{-1} \mathbf{L} \qquad (2.3.14)$$

Making use of the transformation $\mathbf{y} = \mathbf{L}'^{-1}\mathbf{x}$ where $\mathbf{V}^{-1} = \mathbf{LL}'$ and remembering $(1\cdot6\cdot13)$ we can easily deduce that when $\boldsymbol{\tau} = 0$

$$\hat{\boldsymbol{\tau}}'. \left[ w\,\mathbf{Q} + \frac{w'}{k}\mathbf{NB} \right] - w'\,\frac{G^2}{bk} \qquad (2\cdot3\cdot15)$$

has $\chi^2$ distribution with $(v-1)$ degrees of freedom.

4. *Expectations of Sums of Squares in Intrablock Analysis with the Model of Section* 3.

Erom $(2\cdot3\cdot11)$

$$\mathrm{E}\,(G) = bk\mu + \sum_{s=1}^{v} r_s \tau_s$$

$$E(G^2/bk) = V(G/\sqrt{bk}) + [E(G/\sqrt{bk})]^2 = \frac{1}{w'} + \left\{ \sqrt{bk}\,\mu + \frac{1}{\sqrt{bk}}\Sigma r_s\tau_s \right\}^2 \qquad (2\cdot4\cdot1)$$

$$E\,[\text{Total Sum of Squares}] = E\left[\sum_{i=1}^{b}\sum_{j=1}^{k} y^2_{ij} - G^2/bk\right]$$

$$= (bk-1)\,\sigma_1^2 + k\,(b-1)\,\sigma_2^2 + \sum r_s\tau_s{}^2 - \frac{(\Sigma r_s\tau_s)^2}{bk} \quad (2.4.2)$$

$$E\,(T_s) = r_s\,(\mu+\tau_s),\ E\,(T_s^2) = r_s\,(\sigma_1^2+\sigma_2^2) + r_s^2\,(\mu+\tau_s)^2$$

$$\therefore\ E\left[\sum_{s=1}^{v}\frac{T_s^2}{r_s} - \frac{G^2}{bk}\right] = (v-1)\sigma_1^2 + (v-k)\sigma_2^2 + \Sigma\, r_s\,\tau_s^2 - (\Sigma r_s\,\tau_s)^2/bk \quad (2.4.3)$$

$$E\left[\sum_{i=1}^{b}\frac{B_i^2}{k} - \frac{G^2}{bk}\right] = (b-1)(\sigma_1^2 + k\sigma_2^2) + \frac{1}{k}\sum_{i=1}^{b}(\,n_{1i}\,\tau_1 + \ldots + n_{vi}\,\tau_v)^2 - \frac{(\Sigma r_s\,\tau_s)^2}{bk} \quad (2.4.4)$$

A linear function $e'\mathbf{y} = (\mathbf{e}'_1, \mathbf{e}'_2 \ldots, \mathbf{e}'_b)\mathbf{y}$ where $\mathbf{e}_i' = (e_{i1}, e_{i2} \ldots, e_{ik})$, belonging to error in intrablock analysis also belongs to error in the analysis with recovery of interblock information. For these linear functions, we have also $\mathbf{e}'_i\ \mathbf{E}_{k1} = 0$.

$$V(\mathbf{e}'\ \mathbf{y}) = \mathbf{e}'\ \text{diag}\ (\sigma_1^2\ \mathbf{I}_k + \sigma_2^2\ \mathbf{E}_{kk} \ldots \sigma_1^2\ \mathbf{I}_k + \sigma_2^2\ \mathbf{E}_{kk})\ \mathbf{e} = \sigma_1^2\ \mathbf{e}'\ \mathbf{e} \quad (2.4.5)$$

If $\mathbf{e}^{(1)\prime}\ \mathbf{y}$ and $\mathbf{e}^{(2)\prime}\ \mathbf{y}$ are orthogonal and both belong to error

$$\text{Cov}\ (\mathbf{e}^{(1)\prime}\ \mathbf{y},\ \mathbf{e}^{(2)\prime}\ \mathbf{y}) = \mathbf{e}^{(1)\prime}\ \text{diag}\ (\sigma_1^2\ \mathbf{I}_k + \sigma_2^2\ \mathbf{E}_{kk}, \ldots, \sigma_1^2\ \mathbf{I}_k + \sigma_2^2\ \mathbf{E}_{kk})\ \mathbf{e}^{(2)}$$
$$= \sigma_1^2\ \mathbf{e}^{(1)\prime}\ \mathbf{e}^{(2)} = 0 \quad (2.4.6)$$

Hence

$$\text{E}\ [\text{Error Sum of Squares in Intrablock Analysis}]$$
$$= (\,bk - b - v + t\,)\ \sigma_1^2 \quad (2.4.7)$$

In the case of connected designs, we have the following table of expectations.

TABLE : 2.4.1

*Expectations of constituent sums of squares in the analysis of variance when interblock information is utilised*

| Source | SS | E(SS) | SS | E(SS) | Source |
|---|---|---|---|---|---|
| Blocks (unadj) | ... | $(b-1)(\sigma_1^2+k\sigma_2^2)+{}^1/_k\boldsymbol{\tau}'\mathbf{NN}'\boldsymbol{\tau}-(\Sigma r_s\tau_s)^2/bk$ | $V_2$ | $(b-1)\sigma_1^2+(bk-v)\sigma_2^2$ | Blocks (adj.) |
| Treatments (adj) | ... | $(v-1)\sigma_1^2+\boldsymbol{\tau}'\mathbf{C}\boldsymbol{\tau}$ | | $(v-1)\sigma_1^2+(v-k)\sigma_2^2+\Sigma r_s T_s^{\,2}-\dfrac{(\Sigma r_s\tau_s)}{bk}$ | Treatments (unadj) |
| Error (Intrablock) | $V_1$ | $(bk-b-v+1)\sigma_1^2$ | $V_2$ | $(bk-b-v+1)\sigma_1^2$ | Error (Intrablock) |
| Total | | $(bk-1)\sigma_1^2+k(b-1)\sigma_2^2+(\Sigma r_s\tau_s^{\,2})-\dfrac{(\Sigma r_s\tau_s)^2}{bk}$ | | $(bk-1)\sigma_1^2+k(b-1)\sigma_2^2+\Sigma r_s\tau_s^2-\dfrac{(\Sigma r_s\tau_s)^2}{bk}$ | Total |

$$E(V_1) = (bk - b - v + 1)\,\sigma_1^2,\; E\left[\frac{V_1}{bk - b - v + 1}\right] = \sigma_1^2 \tag{2.4.8}$$

$$E(V_2) = (b-1)\,\sigma_1^2 + (bk-v)\,\sigma_2^2$$

$$E\left[\frac{V_2}{bk-v} - \frac{(b-1)\,V_2}{(bk-b-v+1)\,(bk-v)} = \sigma_2^2\right] \tag{2.4.9}$$

$$E\left[\frac{1}{bk-v}\left(k\,V_2 - \frac{v-k}{bk-v-b+1}\,V_1\right)\right] = \sigma_1^2 + k\,\sigma_2^2 \tag{2.4.10}$$

When the experiment can be arranged in $r$ complete replications, $\frac{v}{k} = \frac{b}{r} = l$ (say). The expected value of the yield in the $q$ th plot of the $j$ th block of the $i$ th replication for fixed $\beta_{ij}$'s is given by

$$E(y_{ijq}) = \mu + \rho_i + \beta_{ij} + \tau_{(ijq)} \tag{2.4.11}$$

$$i = 1, 2, \ldots, r;\; j = 1, 2, \ldots, l;\; q = 1, 2, \ldots, k$$

where $\rho_i$ denotes the effect of the $i$ th replication, $\beta_{ij}$ denotes the effect of the $j$ th block of the $i$ th replication and $\tau_{(ijq)}$ denotes the effect of the treatment applied to the $q$ th plot of the $j$ th block of the $i$ th replication. We assume that $\beta_{ij}$'s are independent normal having mean zero and variance $\sigma^2$ and $y_{ijq}$'s for given $\beta_{ij}$'s are independently distributed with mean given by (2.4.11) and variance $\sigma_1^2$. This leads to

$$E(y_{ijq}) = \mu + \rho_i + \tau_{(ijq)}$$

$$V(y_{ijq}) = \sigma_1^2 + \sigma_2^2$$

$$\mathrm{Cov}(y_{ijq}, y_{i'j'q'}) = \begin{cases} \sigma_2^2 \text{ if } i = i', j = j', q \neq q' \\ 0 \text{ otherwise} \end{cases} \tag{2.4.12}$$

With the above set-up, the following expectations of the constituent sums of squares in the intrablock analysis can be easily obtained.

TABLE 2.4.2.

*Expectations of sums of Squares in Analysis of Variance of Connected Resolvable Designs*

| Source | DF | SS | MSS | E (SS) |
|---|---|---|---|---|
| Replications | $r-1$ | $\frac{1}{v}\sum_{i=1}^{r} R_i^2 - \frac{G^2}{bk}$ | ... | $(r-1(\sigma_1^2 + k\sigma_2^2) + v\left[\sum \rho_i^2 - \frac{(\Sigma\rho_i)^2}{r}\right]$ |
| Blocks within replications (adjusted) | $b-r$ | $(b-r)E_b(a)$ | $E_b(a)$ | $(b-r)\sigma_1^2 + (v-k)(r-1)\sigma_2^2$ |
| Treatments (unadjusted) | $v-1$ | $\sum_{s=1}^{v}\frac{T_s^2}{r} - \frac{G^2}{bk}$ | ... | $(v-1)\sigma_1^2 + (v-k)\sigma_2^2 + r\left[\sum \tau_s^2 - \frac{(\Sigma\tau_s)^2}{v}\right]$ |
| Error | $bk-b-v+1$ | $(bk-b-v+1)E_e$ | $E_e$ | $(bk-b-v+1)\sigma_1^2$ |
| Total | $bk-1$ | $\sum y^2 - \frac{G^2}{bk}$ | ... | $(bk-1)\sigma_1^2 + k(b-1)\sigma_2^2 + r\left(\sum \tau_s^2 - \frac{(\Sigma\tau_s)^2}{v}\right) + v\left(\sum \rho_i^2 - \frac{(\Sigma\rho_i)^2}{r}\right)$ |

$R_i$ Total of the $i$ th replication.
$T_s$ Total of $s$ th treatment.
$G$ Grand total.

$$E_b(a) = E\,\text{st}\left[\sigma_1^2 + \frac{(v-k)\ (r-1)}{b-r}\,\sigma_2^2\right]$$

$$= E\,\text{st}\left[\sigma_1^2 + \frac{k\,(r-1)}{r}\,\sigma_2^2\right], \quad \because \frac{v-k}{k} = \frac{b-r}{r}$$

$$\therefore E\,\text{st}\left[\sigma_1^2 + k\,\sigma_2^2\right] = \frac{r\,E_b\,(a) - E_e}{r-1} \qquad (2.2.13)$$

5. *Total and Partial Confounding:* Let there be $m\,k$ treatments and let us divide them into $m$ sets os $k$ each. We assign to a block all the treatments of a set and there are $r$ replications. In this case from (2.1.14)

$$\mathbf{C} = \text{diag}\left[r\,\mathbf{I} - \frac{r}{k}\,\mathbf{E}_{kk},\ \ r\,\mathbf{I} - \frac{r}{k}\,\mathbf{E}_{kk},\ \ldots\ r\,\mathbf{I} - \frac{r}{k}\,\mathbf{E}_{kk}\right] \qquad (2.5.1)$$

$\mathbf{L}'\,\tau$ is estimable if rank $\mathbf{C}$ = rank $(\mathbf{C}, \mathbf{L})$. Rank $C = m\,(k-1)$ and every intraset contrast is estimable. All interset contrasts are not estimable and they are said to be confounded with the blocks. In the notation of Section I, the adjusted treatment sum of squares is

$$\frac{1}{r}\,\mathbf{T}'\,\mathbf{T} - \frac{1}{k^2 r}\,\mathbf{B}'\,\mathbf{N}'\,\mathbf{N}\,\mathbf{B} \qquad (2.5.2)$$

and has $m\,(k-1)$ degrees of freedom. The unadjusted Block Sum of Squares carry $mr-1$ degree of freedom. Subtracting these two from the total sum of squares we get the Error Sum of Squares carrying $m\,(k-1)\,(r-1)$ degrees of freedom. In order that we get an estimate of error $r$ must be greater than unity.

We may, however, not confound the same treatment contrasts in all the $r$ replications. Let us take up the case where the experiment is laid out in such a manner that any intraset treatment contrast in the $i$ th replication is in the $i'$ th replication either confounded (built up of interset contrasts) or estimable (built up of intraset contrasts). Let the $v = mk$ treatments be numbered as $1, 2, \ldots, v$; let the total yield of the $s$ th treatment be denoted by $T_s$, the treatment applied to the $q$ th plot of the $j$ th block of the $i$ th replication by $\tau_{(ijq)}$ and the corresponding yield by $y_{ijq}$. Let $y_{i(s)}$ denote the yield of the $s$ th treatment in the $i$ th replication and $B_{ij}$ the total of the $j$ th block of the $i$ th replication. Let $\mathbf{L}'\,\boldsymbol{\tau} = l_1\,\tau_1 + l_2\,\tau_2 + \ldots + l_v\,\tau_v$ be estimable in the 1st, 2nd, ..., $r_1$, th replications and confound in $(r_1+1)$ – th, $(r_1+2)$ th, ..., $r$ th replications. Let further $E\,(y_{ijq}) = \mu + \rho_i + \beta_{ij} + T_{(ijq)}$ and $V\,(y_{(ijq)}) = \sigma^2$, $\text{Cov}\,(y_{ijq}, y_{i'j'q'}) = 0$.

Then

$$\mathbf{L}'\,\boldsymbol{\tau} = l_1\,\tau_1 + l_2\,\tau_2 + \ldots, + l_v\,\tau_v$$

$$= \sum_{j=1}^{m}\ \sum_{q=1}^{k}\ l_{ijq}\ \tau_{(ijq)} \qquad (2.5.3)$$

where $(i)$ $l_{ij1} + l_{ij2} + \ldots + l_{ijk} = 0$, for $i = l, 2, \ldots, r$,

and $(ii)$ $l_{ij1} = l_{ij2} = \ldots = l_{ijk} = d_{ij}$

$$\sum_{j=1}^{m} d_{ij} = 0, \; i = r_1 + 1, r_1 + 2, \ldots, r$$

$$\text{Now } \frac{1}{r_1} \sum_{i=1}^{r_1} \sum_{j=1}^{m} \sum_{q=1}^{k} l_{ijq} \, y_{ijq} = \frac{1}{r_1} \sum_{i=1}^{r_1} \sum_{s=1}^{v} l_s \, y_{i(s)}$$

$$E \frac{1}{r_1} \left[ \sum_{i=1}^{r_1} \sum_{j=1}^{m} \sum_{q=1}^{k} l_{ijq} \, y_{ijq} \right] = \frac{1}{r_1} \sum_{i=1}^{r_1} \sum_{j=1}^{m} \sum_{q=1}^{k} l_{ijq} \, \tau_{(ijq)} = \frac{1}{r_1} r_1 \, \mathbf{L}' \boldsymbol{\tau}$$

$$= \mathbf{L}' \boldsymbol{\tau} \quad (2.5.4)$$

Also

$$\frac{1}{r_1} \sum_{i=1}^{r_1} \sum_{j=1}^{m} \sum_{q=1}^{k} l_{ijq} \, y_{ijq} = \frac{1}{r_1} \sum_{i=1}^{r} \sum_{j=1}^{m} \sum_{q=1}^{k} l_{ijq} \, y_{ijq} - \frac{1}{r_1} \sum_{i=r_1+1}^{r} \sum_{j=1}^{m} \sum_{q=1}^{k} l_{ijq} \, y_{ijq}$$

$$= \frac{l_1 T_1 + l_2 T_2 + \ldots + l_v T_v}{r_1} - \frac{1}{r_1} \sum_{i=r_1+1}^{r} \sum_{j=1}^{m} d_{ij} \, B_{ij} \quad (2.5.5)$$

is a linear function of block totals and treatment totals. Hence it is the best unbiased linear estimate of $\mathbf{L}' \boldsymbol{\tau}$. The variance of this estimate is

$$\frac{\sigma^2}{r_1^2} \sum_{i=1}^{r_1} \sum_{j=1}^{v} l_j^2 = \frac{\sigma^2}{r_1} \sum_{j=1}^{v} l_j^2 \quad (2.5.6)$$

Had $\mathbf{L}' \boldsymbol{\tau}$ been estimable in all the replication, its variance would have been

$$\frac{\sigma^2}{r^2} \sum_{i=1}^{r} \sum_{j=1}^{v} l_j^2 = \frac{\sigma^2}{r} \sum_{j=1}^{v} l_j^2 \quad (2.5.7)$$

Hence if $\mathbf{L}' \boldsymbol{\tau}$ is partially confounded, the variance of the estimate is $\dfrac{r}{r_1}$ times the variance that would be obtained if it had been unconfounded in all the replication. It is also to be noted that the best unbiased linear estimate of an estimable treatment contrast is obtained from the corresponding linear function of the yields from those replications of the experiment in which it is unconfounded.

## Exercises to Chapter II

1. It is proposed to test seven treatments $A$, $B$, $C$, $D$, $E$, $F$ & $G$ in blocks of three plots according to one of the following plans :

| Plan | Block 1 | Block 2 | Block 3 | Block 4 | Block 5 | Block 6 | Block 7 | Block 8 |
|---|---|---|---|---|---|---|---|---|
| $I$ | $A,B,C$ | $B,F,D$ | $C,D,G$ | $D,A,E$ | $E,C,F$ | $F,G,A$ | $G,E,B$ | |
| $II$ | $A,B,C$ | $B,C,D$ | $C,D,A$ | $D,A,B$ | $D,F,G$ | $F,G,E$ | $G,E,D$ | $E,D,F$ |
| $III$ | $A,B,C$ | $A,C,D$ | $A,D,E$ | $A,E,F$ | $A,F,G$ | $A,G,B$ | | |

Which plan would you recommend and why ?

2. From the incidence matrix of a design given below, obtain ( $i$ ) the estimable treatment contrasts and ( $ii$ ) the degrees of freedom associated with the adjusted treatment and adjusted block sum of squares.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

3. In a connected design, obtain an expression for the average variance of the estimates of all elementary treatment contrasts in terms of the non-zero eigen values of C.

4. Suppose that the elementary treatment contrast $\tau_s - \tau_{s'}$ is estimable. Show that the variance of its best unbiased linear estimate will lie between $2\sigma^2/\lambda_{max}$ and $2\sigma^2/\lambda_{min}$ where $\lambda_{max}$ and $\lambda_{min}$ are the maximum and minimum non-zero eigen values of C.

5. Show that a necessary and sufficient condition in order that all elementary treatment contrasts may be estimated with the same precision is that C has ( $v-1$ ) equal non-zero eigen values.

6. If, in any design, every treatment is replicated the same number of times, the total relative loss of information is one less than the average number of blocks per replication.

Show further that the result is valid for designs in which heterogeneity is eliminated in two directions.

7. An incomplete block design is to be used for two-way elimination of heterogeneity. State the characteristics of such a design, if all treatment contrasts are to be estimated with the same accuracy.

8. Using the notation of section 2, show that a necessary and sufficient condition that estimable treatment contrasts are orthogonal to both estimable row contrasts and estimable column contrasts is

$$\mathbf{F}\ \text{diag}\,(1/r_1, 1/r_2, \ldots, 1/r_v)\ \ \mathbf{L} = \mathbf{O} \text{ and } \mathbf{F}\ \text{diag}\,(1/r_1, 1/r_2, \ldots, 1/r_v)\,\mathbf{M} = \mathbf{O}.$$

9. If $k\,m$ treatments are divided into $m$ sets of $k$ each and if treatments of a set are assigned to $k$-plot blocks and if there be $r$ replications, show that the design is such that the adjusted block effects and adjusted treatment effects are mutually orthogonal.

10. Below is given the plan of a design, there being 12 treatment and 3 replications. The numbers in the body of the table indicate treatments. Identify treatment contrasts which are partially confounded with the blocks and indicate the loss of information on them due to confounding.

| Replication 1 | | Replicatioa 2 | | Replication 3 | |
|---|---|---|---|---|---|
| Block 1 | Block 2 | Block 1 | Block 2 | Block 1 | Block 2 |
| 2,3,5,3,9,12 | 1,4,6,7,10,11 | 1,4,6,7,9,12 | 2,3,5,8.10,11 | 1,4,5,8,10,11 | 2,3,6,7,9,12 |

11. Form an analysis of variance appropriate to the design whose incidence matrix $\mathbf{N} = 2\mathbf{E}_{vb}$ and compare it with that of a design whose incidence matrix is $\mathbf{E}_{vb}$.

12. Show that in each of the following designs, the rank of $\mathbf{C}$ is $v-1$ and there is only one non-zero eigen value. Hence find the variance of best unbiased linear estimate of an elementary treatment contrast

$$(i)\ \ \mathbf{N} = \mathbf{E}_{vb} \qquad (ii)\ \ \mathbf{C} = a\,\mathbf{I} - (a/v)\,\mathbf{E}_{vv}$$

13. Defining $Q_s = T_s - Q_s - (G/bk)$ and other symbols as in section 1 obtain the following in the analysis with recovery of interblock information:

$(i)$ $V(Q_s)$, $(ii)$ $V(Q'_s)$, $(iii)$ Cov $(Q_s, Q_{s'})$, $(iv)$ Cov $(Q_s, Q'_s)$ and $(v)$ Cov $(Q'_s, Q'_{s'})$.

Show that in the set up of section *3*, $\mu + \tau_s$ is an estimable parametric function.

14. $\mathbf{Q}$ and $\mathbf{B}$ having the meaning of section I, establish that in the analysis with recovery of interblock information the dispersion matrix $\mathbf{Q}$ and $\mathbf{B}$ is

$$\sigma^2 \begin{bmatrix} \mathbf{C} & \mathbf{O} \\ \mathbf{O} & \text{diag}\,(k_1, k_2, \ldots, k_b) \end{bmatrix} + \sigma'^2 \begin{bmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \text{diag}\,(k_1^2, k_2^2, \ldots, k_b^2) \end{bmatrix}$$

15. Four fertilisers $A$, $B$, $C$, $D$ each at two levels — single dose and a double dose, and a control (no fertiliser) are tested in $r$ block of 10 plots each. Eight plots are randomly assigned to the fertilisers and two to the control

in each block. Find estimates of the treatment effects and give the analysis of variance. Obtain the variances of estimated comparisons between any two treatment effects. Break up the sum of squares and degrees of freedom due to treatments into (*i*) between fertilisers (single dose), (*ii*) between fertilisers (double dose), (*iii*) between doses.

16. Show that the best unbiased linear estimate of an estimable treatment contrast $\mathbf{L}'\boldsymbol{\tau}$ is given by $\mathbf{d}'\,\mathbf{Q} = d_1 Q_1 + d_2 Q_2 + \ldots + d_v Q_v$ where $\mathbf{C}\,\mathbf{d} = \mathbf{L}$ and the variance of this estimate is $\sigma^2\,\mathbf{d}'\,\mathbf{C}\,\mathbf{d} = \sigma^2\,\mathbf{d}'\,\mathbf{L}$. If $\mathbf{C} = a\mathbf{I} - \frac{a}{v}\,\mathbf{E}_{vv}$, show that the elementary contrast $\tau_s - \tau_{s'}$ has for its best estimate $(Q_s - Q_{s'})/a$ with variance $2\,\sigma^2/a$.

17. Show that the equations for obtaining $\tau_s$ in the combined inter and intra block analysis can be obtained from the corresponding equations in intrablock analysis if we replace $Q_s$, $r_s$ and $\lambda_{ss'}$ by $P_s$, $R_s$ and $\Lambda_{ss'}$ where $P_s = w\,Q_s + w'\,Q_s$, $R_s = r_s\,[w + w'/(k-1)]$, $\Lambda_{ss'} = (w - w')\lambda_{ss'}$ and set in addition $\sum_{s=1}^{v} r_s\,\tau_s = 0$

18. In the intrablock analysis, find the expected values of (*i*) adjusted treatment sum of squares and (*ii*) adjusted block sum of squares.

19. $v = s_1 \times s_2$ treatments $v_{ij}$, $(i = 1, 2, \ldots, s_1;\ j = 1, 2, \ldots, s_2)$ are arranged in $b$ blocks with $k$ plots each such that every treatment is replicated $r$ times and $v_{ij}$ and $v_{i'j'}$ occur together in $\lambda_{k_1 k_2}$ plots where $k_1 = 0$ or $1$ according as $i = i'$ or $i \neq i'$ and $k_2 = 0$ or $1$ according as $j = j'$ or $j \neq j'$. From the relation $(\mathbf{E}_{1v}\,\mathbf{N})\,\mathbf{N}' = \mathbf{E}_{1v}\,(\mathbf{N}\,\mathbf{N}')$, establish

$$(s_1 - 1)\,\lambda_{10} + (s_2 - 1)\,\lambda_{01} + (s_1 - 1)(s_2 - 1)\,\lambda_{11} = r\,(k - 1)$$

Write down the normal equations in the analysis of such designs and obtain variances of best unbiased linear estimates of comparisons of the form

$$(i)\ \sum_{j=1}^{s_2} v_{ij} - \sum_{j=1}^{s_2} v_{i'j}\,,\ i \neq i' \quad (ii)\ \sum_{i=1}^{s_1} v_{ij} - \sum_{i=1}^{s_1} v_{ij'}\,,\ j \neq j'$$

and (*iii*) $v_{ij} + v_{i'j'} - v_{ij'} - v_{i'j}$

20. Let there be $N = n_1 + n_2 + \ldots, + n_k$ plots and let $k$ treatments be assigned at random to these $N$ plots, $n_i$ plots having the $i$ th treatment. The yields are independent normal having a common variance $\sigma^2$ and expected values given by the effect of the treatment applied to the plot. Test the equality of the treatment effects and show that the test is most sensitive when $n_1 = n_2 = \ldots, n_k = N/k$.

21. Show that in a connected design, $Q_i + r_i G/n$, $i = 1, 2, \ldots, v$ are linearly independent. Hence establish that

( $i$ ) $\mathbf{C} + \mathbf{r}\,\mathbf{r}'/n$ is non-singular

( $ii$ ) $(\mathbf{C} + \mathbf{r}\,\mathbf{r}'/n)^{-1}\,\mathbf{r} = \mathbf{E}_{v1}$

where $\mathbf{r} = \{r_1, r_2, \ldots, r_v\}$, $n = r_1 + r_2 + \ldots, r_v$.

22. Show that in a connected design

$$\hat{\mu} = G/n,\ \hat{\boldsymbol{\tau}} = (\mathbf{C} + \mathbf{r}\,\mathbf{r}'/n)^{-1}\,\mathbf{Q}$$

$$\hat{\boldsymbol{\alpha}} = \text{diag}\ (1/k_1, 1/k_2, \ldots, 1/k_b)\,\mathbf{B} - (G/n)\,\mathbf{E}_{b1} - \text{diag}\ (1/k_1, \ldots, 1/k_b),\ \ \mathbf{N}'\,(\mathbf{c} + \mathbf{r}\,\mathbf{r}'/n)^{-1}\,\mathbf{Q}$$

will satisfy the normal equations in intrablock analysis. Show further that the dispersion matrix of $\hat{\boldsymbol{\tau}}$ is $\sigma^2\,[(\mathbf{c} + \mathbf{r}\,\mathbf{r}'/n)^{-1} - \mathbf{E}_{vv}/n]$ and the variance of the best estimate of a treatment contrast $\mathbf{L}'\,\boldsymbol{\tau}$ is $\sigma^2\,\mathbf{L}'\left(\mathbf{c} + \dfrac{\mathbf{r}\mathbf{r}'}{r}\right)^{-1}\mathbf{L}$

23. ( $i$ ) When $T_1$ and $T_2$ are independent, show that the linear function of $T_1$ and $T_2$ having least variance is

$$\left[\sigma^2_{T_2}\,T_1 + \sigma^2_{T_1}\cdot T_2\right] \Big/ \left[\sigma^2_{T_1} + \sigma^2_{T_2}\right]$$

( $ii$ ) In a connected design of equal block-size $k$ where further $\mathbf{N}\,\mathbf{N}'$ is non-singular, show that the intrablock, interblock and combined inter and intra-block estimates of the treatment contrast $\mathbf{L}'\,\boldsymbol{\tau}$ are respectively

$$\mathbf{L}'\left[\mathbf{c} + \frac{\mathbf{r}\mathbf{r}'}{n}\right]^{-1}\mathbf{Q},\quad \mathbf{L}'\,(\mathbf{N}\mathbf{N}')^{-1}\,\mathbf{N}\,\mathbf{B}\ \text{and}$$

$$\mathbf{L}'\left[w\,\mathbf{C} + \frac{w'}{k}\,\mathbf{N}\mathbf{N}'\right]^{-1}\left[w\,\mathbf{Q} + \frac{w'}{k}\,\mathbf{N}\,\mathbf{B}\right]$$

Use ( $i$ ) to obtain the linear combination of the first two estimates having least variance and show that it is in general different from the third estimate. Obtain situations where these two are equal.

24. Show that

$$(i)\qquad \begin{vmatrix} \mathbf{C} & -\mathbf{N}' \\ \mathbf{N}' & \text{diag}\,(k_1, \ldots, k_b) \end{vmatrix} = k_1\,k_2 \ldots k_b\,r_1\,r_2 \ldots r_v$$

$$(ii)\qquad \left[w\mathbf{C} + \frac{w'}{k}\,\mathbf{N}\mathbf{N}'\right]^{-1}\mathbf{r} = \frac{1}{w'}\,\mathbf{E}_{v1},\quad \mathbf{r} = \{r_1, r_2, \ldots, r_v\}$$

When $r_1 = r_2 = \ldots = r_v = r$, show that the average variance of all elementary treatment contrasts with recovery of interblock information is

$$2\left[\text{Trace}\left[w\mathbf{C} + \frac{w'}{k}\,\mathbf{N}\,\mathbf{N}'\right]^{-1} - \frac{1}{w'r}\right] \Big/ (v-1)$$

CHAPTER III

# STANDARD DESIGNS

1. *Randomised Block Design.* Here we have $bk$ plots divided into $b$ blocks of $k$ plots each and each of the $v = k$ treatments is assigned at random to one of these plots in each block. Evidently for this design

$$\mathbf{N} = \mathbf{E}_{vb} \tag{3.1.1}$$

$$\mathbf{C} = b\left[\mathbf{I} - \frac{1}{b}\,\mathbf{E}_{vc}\right],\ \mathbf{D} = k\left[\mathbf{I} - \frac{1}{b}\,\mathbf{E}_{bb}\right] \tag{3.1.2}$$

$$\mathbf{Q} = \mathbf{T} - \frac{G}{k}\,\mathbf{E}_{v1} \tag{3.1.3.}$$

$$\hat{\boldsymbol{\tau}} = \frac{1}{b}\,\mathbf{T}, \qquad \hat{\boldsymbol{\alpha}} = \frac{1}{b}\,\mathbf{B} \tag{3.1.4}$$

Condition (2.1.26) is satisfied and the adjusted treatment and block yields are orthogonal.

TABLE 3.1.1

*Analysis of Variance Table for Randomised Block Design*

| Source | DF | SS | E(SS) |
|---|---|---|---|
| Blocks | $b-1$ | $\frac{1}{v}\,\mathbf{B}'\mathbf{B} - \frac{G^2}{bk}$, | $(b-1)\,\sigma^2 + v\boldsymbol{\alpha}'\left[\mathbf{I} - \frac{1}{b}\,\mathbf{E}_{bb}\right]\boldsymbol{\alpha}$ |
| Treatments | $v-1$ | $\frac{1}{b}\,\mathbf{T}'\mathbf{T} - \frac{G^2}{bk}$, | $(v-1)\,\sigma^2 + b\,\boldsymbol{\tau}'\left[\mathbf{I} - \frac{1}{v}\,\mathbf{E}_{vv}\right]\boldsymbol{\tau}$ |
| Error | $(b-1)(v-1)$ | $\mathbf{y}'\mathbf{y} - \frac{1}{b}\,\mathbf{T}'\mathbf{T} - \frac{1}{v}\,\mathbf{B}'\mathbf{B} + \frac{G^2}{bk}$, | $(b-1)(v-1)\sigma^2$ |
| Total | $bv-1$ | $\mathbf{y}'\mathbf{y} - \frac{G^2}{bk}$ | |

2. *Latin Square Design.* It is an arrangement of $u^2$ plots of $u$ rows and $u$ columns in which $u$ treatments are assigned in such a way that each treatment occurs once and only once in each row and in each column. Adopting the notations of Section 2 of Chapter 2, we have

$$L = M = E_{uu}$$

rank $\mathbf{F}$ = rank $[\,u\,\mathbf{I} - \mathbf{E}_{uu}\,] = u - 1$ and every treatment contrast is estimable. Similarly it can be seen that all block contrasts and all column contrasts are estimable. The best estimates of estimable row, column and treatment contrasts

$$\mathbf{R} - \frac{G}{u}\,\mathbf{E}_{u1},\ \mathbf{C} - \frac{G}{u}\,\mathbf{E}_{u1} \text{ and } \mathbf{T} - \frac{G}{u}\,\mathbf{E}_{u1} \qquad (3\cdot2\cdot1)$$

are mutually orthogonal.

Equation $(2\cdot2\cdot7)$ now becomes

$$\mathbf{T} - \frac{G}{u}\,\mathbf{E}_{u1} = \left[ u\,\mathbf{I} - \mathbf{E}_{uu} \right] \hat{\boldsymbol{\tau}} \qquad (3\cdot2\cdot2)$$

And a solution of $(3\cdot2\cdot2)$ can be taken as

$$\hat{\boldsymbol{\tau}} = \frac{1}{u}\,\mathbf{T} \qquad (3\cdot2\cdot3)$$

Sometimes when the experiment is over, one is interested in whether the additional column classification was worthwhile or not. By equating E(SS) for columns, Treatments and Error in Table 3·2·1 with those for Treatments and Error in Table 3·1·1, we get

$$u\,(u-1)\,\sigma_L^2 + u\,\boldsymbol{\beta}' \left[ \mathbf{I} - \frac{1}{u}\,\mathbf{E} \right] \boldsymbol{\beta} + u\,\boldsymbol{\tau}' \left[ \mathbf{I} - \frac{1}{u}\,\mathbf{E}_{uu} \right] \boldsymbol{\tau} = u\,(u-1)\,\sigma_{RB}^2$$

$$+ u\,\boldsymbol{\tau}' \left[ \mathbf{I} - \frac{1}{u}\,\mathbf{E}_{uu} \right] \boldsymbol{\tau}$$

$$\text{Hence Est } \sigma_{RB}^2 = \frac{(u-1)^2\,\mathrm{E}_L + \text{Column SS in Latin Square}}{u\,(u-1)} \qquad (3\cdot2\cdot4)$$

Comparison of (3.2.4) with $E_L$ gives one an idea as to whether column classification has led to increased precision or not.

3. *Graeco-Latin and Higher Order Squares.* Consider $m$ set of symbols

$$A_{11}, A_{12}, \ldots, A_{1u}$$
$$A_{21}, A_{22}, \ldots, A_{2u}$$
$$A_{m1}, A_{m2}, \ldots, A_{mu}$$

TABLE 3·2·1

*Analysis of Variance of Latin Square Design*

| Source | DF | SS | MSS | E(SS) |
|---|---|---|---|---|
| Rows | $u-1$ | $\frac{1}{u}\mathbf{R}'\mathbf{R} - \frac{G^2}{u^2}$ | ... | $(u-1)\sigma^2 + u\boldsymbol{\alpha}'\left[\mathbf{I} - \frac{1}{u}\mathbf{E}_{uu}\right]\boldsymbol{\alpha}$ |
| Columns | $u-1$ | $\frac{1}{u}\mathbf{C}'\mathbf{C} - \frac{G^2}{u^2}$ | ... | $(u-1)\sigma^2 + u\boldsymbol{\beta}'\left[\mathbf{I} - \frac{1}{u}\mathbf{E}_{uu}\right]\boldsymbol{\beta}$ |
| Treatments | $u-1$ | $\frac{1}{n}\mathbf{T}'\mathbf{T} - \frac{G^2}{u^2}$ | ... | $(u-1)\sigma^2 + u\boldsymbol{\tau}'\left[\mathbf{I} - \frac{1}{u}\mathbf{E}_{uu}\right]\boldsymbol{\tau}$ |
| Error | $(u-1)(u-2)$ | — | $\mathrm{E}_L$ | $(u-1)(u-2)\sigma^2$ |
| Total | $u^2-1$ | $\mathbf{y}'\mathbf{y} - \frac{G^2}{u^2}$ | | |

where $m < u$. Let there be $u^2$ cells arranged in $u$ rows and $u$ columns. Then an arrangement of these $m\ u$ symbols in the $u^2$-plots will be called an orthogonal Latin Square of order $m$ if

(*i*) in every cell there is a combination of one $A_1$-symbol, one $A_2$-symbol, ..., one $A_m$-symbol, the particular combination in the cell in the $i$ th row and $j$ th column being donated by

$A_{1(ij)}\ A_{2(ij)} \cdots A_{m(ij)}$

(*ii*) $A_{t(11)}\ A_{t(12)} \cdots A_{t(1u)}$

$A_{t(21)}\ A_{t(22)} \cdots A_{t(2n)}$

$A_{t(u1)}\ A_{t(u2)} \cdots A_{t(uu)}$

constitutes a Latin Square, $t = 1, 2, \ldots, m$

(*iii*) any combination of an $A_t$-symbol and an $A_{t'}$ - symbol $(t \neq t')$ appears in one and only one cell.

In the special case $m = 2$, the arrangement is known as Graeco-Latin Square. When $u$ is a power of a prime, such arrangements are always possible provided $m \leqslant (u-1)$. Let $y_{ij}$ be the yield corresponding to the $(ij)$-th cell, $\tau_{ti}$ the effect of $A_{ti}$ $(t = 1, 2, \ldots, m;\ i = 1, 1, \ldots, u)$, $\alpha_i$ effect of the $i$ th row, $\beta_j$ effect of $j$ th column and $\mu$, the general effect. Using the same notations as in the previous section and denoting by $T_{ti}$ the total yield corresponding to $A_{ti}$, we have the following set-up

$$E(y_{ij}) = \mu + \alpha_i + \beta_j + \tau_{1(ij)} + \tau_{2(ij)} + \cdots + \tau_{m(ij)} \tag{3.3.1}$$

and the following normal equations :

$$G = u^2\hat{\mu} + u\sum_{i=1}^{b}\hat{\alpha}_i + u\sum_{j=1}^{b}\hat{\beta}_j + u\sum_{t=1}^{m}\sum_{i=1}^{u}\hat{\tau}_{ti}$$

$$R_i = u\hat{\mu} + u\hat{\alpha}_i + \sum\hat{\beta}_j + \sum\sum\hat{\tau}_{ti}\ ,\ i = 1, 1, \ldots, u$$

$$C_j = u\hat{\mu} + \sum\hat{\alpha}_i + u\hat{\beta}_j + \sum\sum\hat{\tau}_{ti}\ ,\ j = 1, 2, \ldots, u$$

$$T_{ti} = u\hat{\mu} + \sum\hat{\alpha}_i + \sum\hat{\beta}_j + u\hat{\tau}_{ti} + u\sum_{t'=1}\sum_{i'=1}^{u}\hat{\tau}_{t'i'}$$

$$t = 1, 2, \ldots, m;\ i = 1, 2, \ldots, u. \tag{3.3.2}$$

The number of independent equations in (3·3·2) is $(m+2)(u-1)+1$.

A set of solutions is given by

$$\hat{\mu} = \frac{G}{u^2},\ \hat{\alpha}_i = \frac{R_i}{u} - \frac{G}{u^2},\ \hat{\beta}_j = \frac{C_j}{u} - \frac{G}{u^2} \text{ and } \hat{\tau}_{ti} = \frac{T_{ti}}{u} - \frac{G}{u^2}$$

The sum of squares due to regression when all constants are fitted with $d.f. = (m+2)(u-1)+1$ is

$$\frac{G^2}{u^2} + \sum_{i=1}^{u}\left[\frac{R_i^2}{u} - \frac{G^2}{u^2}\right] + \sum_{j=1}^{u}\left[\frac{C_j^2}{u} - \frac{G^2}{u^2}\right] + \sum_{t=1}^{m}\left[\sum_{i=1}^{u}\frac{T_{ti}^2}{u} - \frac{G^2}{u^2}\right]$$

Easy calculations will give the following sum of squares due to regression when $\tau_{mi}\ (i = 1, 1, \ldots, u)$ are dropped

$$\frac{G^2}{u^2} + \sum_{i=1}^{u}\left[\frac{R_i^2}{u} - \frac{G^2}{u}\right] + \sum_{j-1}^{u}\left[\frac{C_j^2}{u} - \frac{G^2}{u}\right] + \sum_{t=1}^{m-1}\left[\sum_{i=1}^{u}\frac{T_{ti}^2}{u} - \frac{G^2}{u^2}\right] \tag{3.3.5}$$

with $d.f. = (m+1)(u-1)+1$. The difference between (3·3·4) and (3.3.5)

$$\sum_{i-1}^{u}\frac{T^2{}_{mi}}{u} - \frac{G^2}{u^2} \tag{3.3.6}$$

is distributed as $\chi^2\ \sigma^2$ with d.f. $= u-1$ when $\tau_{m1} = \ldots = \tau_{mu} = 0$.

TABLE 3.3.1

*Analysis of Variance Table for Orthogonal Latin Square Design of $m^{th}$ Order*

| Source | DF | SS | E(SS) |
|---|---|---|---|
| Rows | $u-1$ | $\sum_{i=1}^{u}\frac{R_i^2}{u} - G^2/u^2$ | $(u-1)\sigma^2 + u\left[\sum\alpha_i^2 - \frac{(\Sigma\alpha_i)^2}{u}\right]$ |
| Columns | $u-1$ | $\sum_{j=1}^{u} C_j^2/u - G^2/u^2$ | $(u-1)\sigma^2 + u\left[\sum\beta_j^2 - \frac{(\Sigma\beta_j)^2}{u}\right]$ |
| $A_1$ symbols | $u-1$ | $\sum_{i=1}^{u} T_{1i}^2/u - G^2/u^2$ | $(u-1)\sigma^2 + u\left[\sum\tau_{1i}^2 - \frac{(\Sigma\tau_{1i})^2}{u}\right]$ |
| $A_2$ symbols | $u-1$ | $\sum_{i=1}^{u} T_{2i}^2/u - G^2/u^2$ | $(u-1)\sigma^2 + u\left[\sum\tau_{2i}^2 - \frac{(\Sigma\tau_{2i}^2)^2}{u}\right]$ |
| ... | ... | ... | ... ... ... |
| $A_m$ symbols | $u-1$ | $\sum_{i=1}^{u} T_{mi}^2/u - G^2/u^2$ | $(u-1)\sigma^2 + u\left[\sum\tau_{mi}^2 - \frac{(\Sigma\tau_{mi})^2}{u}\right]$ |
| Error | $(i-1)(u-1-m)$ | ... | $(u-1)(u-1-m)\sigma^2$ |
| Total | $u^2-1$ | $\sum y^2 - \frac{G^2}{u^2}$ | |

4. *Cross-over Design.* From the point of view of analysis, this design is nothing but $p$ replications of an $s \times s$ Latin Square Design. Adopting the notations of Section 2 of Chapter 2, we have in this case $v = s$, $u = s$, $u' = sp$, $l_{ij} = p$, $m_{ij'} = 1$,

$\mathbf{L} = p\ \mathbf{E}_{ss}$, $\mathbf{M} = \mathbf{E}_{s(sm)}$

$$\mathbf{F} = sp\ \mathbf{I} - p\ \mathbf{E}_{ss},\ \mathbf{Q} = \mathbf{T} - \frac{G}{s}\mathbf{E}_{s1},\ \hat{\boldsymbol{\tau}} = \frac{\mathbf{T}}{sp} - \frac{G}{s^2p}\mathbf{E}_{s1}$$

$\hat{\boldsymbol{\tau}}'\mathbf{Q} = \dfrac{\mathbf{T}'\mathbf{T}}{sp} - \dfrac{G^2}{s^2p}$ · Rank $\mathbf{F} = s - 1$ and hence every treatment contrast is estimable. The analysis of variance table can now be obtained from Table 2.2.1.

5. *Balanced Incomplete Block Design.* (B I B D). An arrangement of $v$ treatments in $b$ blocks of $k$ plots each, $(k < v)$, is known as B I B D if every variety occurs once and only once in $r$ blocks and any two varieties occur together in $\lambda$ blocks. Let $\mathbf{N}$ be the incidence matrix of this design. Then

$$\mathbf{E}_{1v}\mathbf{N}\mathbf{E}_{b1} = (bk) = \mathbf{E}_{1b}\mathbf{N}'\mathbf{E}_{v1} = (vr) \quad \cdots \tag{3.5.1}$$

$$\mathbf{N}\mathbf{N}'\mathbf{E}_{v1} = [r + \lambda(v-1)]\,\mathbf{E}_{v1} = \mathbf{N}(\mathbf{N}'\mathbf{E}_{v1}) = (rk)\,\mathbf{E}_{v1} \quad \cdots \tag{3.5.2}$$

From (3.5.1) and (3.5.2) we get

$$bk = vr \tag{3.5.3}$$

$$\lambda(v - ) = r(k-1) \tag{3.5.4}$$

We have also

$$\det \mathbf{N}\mathbf{N}' = [r + \lambda(v-1)]\,(r-\lambda)^{v-1} = rk\,(r-\lambda)^{v-1} \neq 0 \tag{3.5.5}$$

Hence rank $\mathbf{N}$ = rank $\mathbf{N}\mathbf{N}' = v \leqslant b$. Thus

$$b \leqslant v \text{ or } r \geqslant k \tag{3.5.6}$$

Also

$$\left(\frac{v}{k} - 1\right)(r-k) \geqslant 0$$

$$b - v - (r-k) \geqslant 0$$

$$b \geqslant v + r - k \tag{3.5.7}$$

When $b = v$, the B I B D is said to be symmetrical. In this case

$$\det \mathbf{N}\mathbf{N}' = [\det \mathbf{N}]^2 = r^2(r-\lambda)^{v-1}$$

$$\therefore \quad \det \mathbf{N} = \pm\, r\,(r-\lambda)^{\frac{v-1}{2}} \tag{3.5.8}$$

det $\mathbf{N}$ is an integer, hence when $v$ is even $(r-\lambda)$ must be a perfect square. For symmetrical B I B D

$$\mathbf{NN'} = (r-\lambda)\mathbf{I} + \lambda\,\mathbf{E}_{vv},\ (\mathbf{NN'})^{-1} = \mathbf{N'}^{-1}\mathbf{N}^{-1} = \frac{1}{(r-\lambda)}\left[\mathbf{I} - \frac{\lambda}{r^2}\,\mathbf{E}_{vv}\right]$$

$$\mathbf{N}^{-1} = \left[\mathbf{N'} - \frac{\lambda}{r}\,\mathbf{E}_{vv}\right] / (r-\lambda)$$

Postmultiplying both sides by $\mathbf{N}$ we get

$$\mathbf{N'N} = (r-\lambda)\,\mathbf{I} + \lambda\,\mathbf{E}_{vv} = \mathbf{NN'} \tag{3.5.9}$$

Hence in the case of a symmetrically balanced incomplete block design, any two blocks have $\lambda$ varieties in common.

If the B I B D is resolvable (i.e. consists of $r$ sets of $\frac{b}{r}$ blocks each, $\frac{b}{r}$ being an integer and each set of blocks contains every treatment once), the inequality (3.5.7.) can be further improved. In this case $\mathbf{N}$ consists of $r$ sets of $\frac{b}{r}$ columns each, where a set of columns is such that 1 occurs once and only once in each row of the set. By adding the 1st, 2nd, ...$\left(\frac{b}{r} - 1\right)$th columns to the $\frac{b}{r}$ th columns of a set, we obtain a column consisting of 1 only. As there are $r$ sets evidently

$$v = \text{Rank } \mathbf{N} \leqslant b - (r-1)$$

$$\therefore b \geqslant v + r - 1 \tag{3.5.10}$$

( $i$ ) Analysis utilising intrablock information only.

$$\mathbf{C} = \frac{v\lambda}{k}\,\mathbf{I} - \frac{\lambda}{k}\,\mathbf{E}_{vv} \tag{3.5.11}$$

$$\mathbf{Q} = \mathbf{C}\,\hat{\tau}$$

is satisfied by $\hat{\tau} = \dfrac{k}{v\lambda}\,\mathbf{Q}$ (3.5.12)

Rank $\mathbf{C} = v - 1$ and sum of squares due to treatments is $\dfrac{k}{v\lambda}\,\mathbf{Q'Q}$. The best unbiased linear estimate of $\tau_s - \tau_{s'}$ is $\dfrac{k}{v\lambda}\,(Q_s - Q_{s'})$

with variance $\sigma^2\,\dfrac{2k}{v\lambda} = \dfrac{2\sigma^2}{r}\,\dfrac{kr}{v\lambda}$.

$$= \frac{2\sigma^2}{r} \cdot \frac{kr\,(k-1)}{v\lambda\,(k-1)} = \frac{2\sigma^2}{r}\;\frac{1 - \dfrac{1}{v}}{1 - \dfrac{1}{k}}$$ . The corresponding

variance in the case of a randomised block design is $2\sigma^2/r$ ( assuming $\sigma^2$ to be the same in both the cases). Hence the efficiency of B I B D is

$$E = \frac{v\lambda}{kr} = \frac{1 - \dfrac{1}{k}}{1 - \dfrac{1}{v}} < 1 \tag{3.5.13.}$$

(*ii*) Analysis with recovery of intrablock information.

$$w\,\mathbf{C} + \frac{w'}{k}\,\mathbf{NN'} = \frac{w\,v\,\lambda + w'\,(r-\lambda)}{k}\,\mathbf{I} + \frac{w'-w}{k}\,\mathbf{E}_{vv}$$

$$\left[w\,\mathbf{C} + \frac{w'}{k}\,\mathbf{NN'}\right]^{-1} = \frac{k}{wv\,\lambda + w'\,(r-\lambda)}\left[\mathbf{I} + \frac{(w-w')}{w'rk}\,\mathbf{E}_{vv}\right]$$

$$\hat{\boldsymbol{\tau}} = \left[w\,\mathbf{C} + \frac{w'}{k}\,\mathbf{NN'}\right]^{-1}\left[w\,\mathbf{Q} + \frac{w'}{k}\,\mathbf{NB}\right]$$

$$= \frac{k}{wv\,\lambda + w'\,(r-\lambda)}\left[w\,\mathbf{Q} + \frac{w'}{k}\,\mathbf{NB} + \frac{(w-w')\,\lambda G}{rk}\,\mathbf{E}_{v1}\right] \tag{3.5.14}$$

Best unbiased linear estimate of a treatment contrast $\mathbf{L}'\boldsymbol{\tau}$ is

$$\frac{k}{wv\,\lambda + w'\,(r-\lambda)}\,\mathbf{L}'\left[w\,\mathbf{Q} + \frac{w'}{k}\,\mathbf{NB}\right] \tag{3.5.15}$$

and its variance by (2.3.14) is

$$\frac{k}{wv\,\lambda + w'\,(r-\lambda)}\,\mathbf{L'L} \tag{3.5.16}$$

The variance of the best estimate of an elementary contrast $\tau_s - \tau_{s'}$ is

$$\frac{2k}{wv\,\lambda + w'\,(r-\lambda)} = \frac{2k(v-1)}{wv\,\lambda(v-1) + w'(r-\lambda)(v-1)}$$
$$= \frac{2}{r}\,\frac{k(v-1)}{v(k-1)w + (v-k)\,w'} \tag{3.5.17}$$

$\dfrac{k(v-1)}{v(k-1)w + (v-k)w'}$ is called the effective variance and can be approximated by $E_e\,[1 + (v-k)\,\nu]$ where $E_e$ is the intrablock error mean square and

$$\nu = \frac{w-w'}{v(k-1)w + (v-k)w'} \tag{3.5.18}$$

The best estimate of the estimable parametric function $\mu + \tau_s$ is

$$\hat{\tau}_s = \frac{T_s}{r} + \frac{\nu}{r}\left[(v-k)T_s - (v-1)B'_s + (k-1)G\right]$$

$$= \frac{1}{r}\left[T_s + \nu W_s\right] \qquad (3.5.19)$$

Denoting by $v_1$ and $v_2$ the error sum of squares and adjusted block sum of squares in the intrablock analysis, we can estimate $w$ and $w'$ by

$$\frac{bk-b-v+1}{v_1} \text{ and } 1/\left\{\frac{v_1}{bk-b-v+1} + \frac{k}{bk-v}\left[V_2 - \frac{V_2\ (b-1)\ V_1}{bk-b-v+1}\right]\right\}$$

respectively. When, however, the B I B D is resolvable, for estimating $w$ and $w'$ we make use of the error sum of squares and adjusted blocks within replications sum of squares and denoting these mean squares by $E_e$ and $E_b(a)$ respectively, we get easily

$$\text{Est. } (w - w') = \frac{E_b(a) - E_e}{E_e\left[rE_b(a) - E_e\right]} \qquad (3.5.20)$$

$$\text{Est. } \nu = \frac{r\left[E_b(a) - E_e\right]}{vr(k-1)E_b(a) + k(b-v-r+1)E_e} \qquad (3.5.21)$$

When the B I B D is symmetrical, the adjusted block sum of squares can be calculated directly. For this expression is

$$\frac{k}{v\lambda}\sum_{s=1}^{v}\left(T_s - \frac{B'_s}{k}\right)^2 + \sum_{i=1}^{b}\frac{B_i^2}{k} - \sum_{s=1}^{b}\frac{T_s^2}{k}$$

$$= \frac{k-\lambda}{kv\lambda}\sum T_s^2 - \frac{2}{v\lambda}\sum T_s B'_s + \frac{k}{v\lambda(k-\lambda)}\sum {B'_s}^2 - \frac{\lambda}{k(k-\lambda)}G^2 \quad . (3.5.22)$$

$$\text{For } \sum_{1}^{v} {B'_s}^2 = (\mathbf{NB})'(\mathbf{NB}) = \mathbf{B}'\mathbf{N}'\mathbf{N}\mathbf{B} = \mathbf{B}'\left[r-\lambda)\mathbf{I} + \lambda\,\mathbf{E}_{vv}\right]\mathbf{B}$$

$$= (r-\lambda)\,\Sigma B_i^2 + \lambda\, G^2 \qquad (3.5.23)$$

Now $W_s = (v-k)T_s - (v-1)B'_s + (k-1)G$

$$\therefore \sum W_s^2 = (v-k)^2\sum T_s^2 - 2(v-k)(v-1)\sum T_s B'_s + (v-1)^2\sum B_s'^2 - v(k-1)^2 G^2 \qquad (3.5.24)$$

$\therefore$ From (3.5.22) and (3.5.24).

$$\text{Adjusted Block SS} = \frac{1}{v\lambda(v-1)(v-k)} \sum_{s=1}^{v} W_s^2 \tag{3.5.22}$$

Sun of squares due to regression when $\tau_1 = \tau_2 = \ldots = \tau_v = 0$

is $\boldsymbol{\tau}'\left(w\,\mathbf{Q} + \frac{w'}{k}\,\mathbf{NB}\right) - \frac{w'G^2}{bk}$

$$= \frac{wv\lambda + w'(r-\lambda)}{kr^2} \sum_{s=1}^{v} (T_s + \nu\, W_s - \overline{T_s + \nu\, W_s})^2 \tag{3.5.26}$$

and this has a $\chi^2$ distribution with d.f. $= v - 1$

Also $w\,(bk - b - v + 1)\,E_e$ has a $\chi^2$ distribution with d.f. $= bk-$ $-b - v + 1$ independent of (3.5.26).

Hence

$$\frac{wv\lambda + w'(r-\lambda)}{kr^2 w E_e (v-1)} \sum_{s=1}^{v} (T_s + \nu\, W_s - \overline{T_s + \nu\, W_s})^2$$

$$= \frac{v(k-1)w + (v-k)\,w'}{kr(v-1)^2 w E_e} \sum_{s=1}^{v} (T_s + \nu\, W_s - \overline{T_s + \nu\, W_s})^2 \tag{3.5.27}$$

has $F$-distribution with $n_1 = v - 1$, $n_2 = bk - b - v + 1$.

Hence

$$\frac{\sum_{s=1}^{v} (T_s + \nu\, W_s - \overline{T_s + \nu\, W_s})^2}{E_e\,[1 + (v-k)\,\nu]} \tag{3.5.28}$$

can be regarded as approximately distributed as an $F$-ratio with $n_1 = v - 1$. $n_2 = bk - b - v + 1$.

6. *Youden Square.* Given a member of subsets $S_1$ $S_2, \ldots, S_n$ of a given set $S$. These subsets will possess a System of Distinct Representatives (SDR) if we can find a set

$$D = [\,a_1, a_2, \ldots, a_n\,]$$

such that $a_1, a_2 \ldots, a_n$ are all distinct and $a_i$ belongs to $S_i$ $(i = 1, 2, \ldots, n)$. We can easily prove by induction that $S_1, S_2, \ldots, S_n$ will possess a S D R if every $t$ of them contain between them at least $t$ distinct elements of $S$, $(t = 1, 2. \ldots, n)$. Consider a symmetrically balanced imcomplete block design arranged in a two-way table in which the blocks are columns. The

sets $S_1, S_2, \ldots, S_v$ constituting the columns are subsets of the treatments $1, 2, \ldots, v$. Between any $t$ of them there will be $t\,k$ elements each one of which can be repetition of the same element at most $k$ times. Hence there must at least be $t$ distinct elements. We can, therefore, find an S D R of these sets which will be nothing but $1, 2, \ldots, v$ arranged in some order. Let these constitute the first row. Consider the residual sets $S_1', S_2', \ldots, S_v'$. Any $t$ of them will contain $t(k-1)$ elements, none being a repetition of the same element more than $(k-1)$ times. Hence there are at least $t$ distinct elements and the subsets possess a S D R which is nothing but $1, 2, \ldots,$ $v$ arranged in some order. Let these constitute the second row. By repeating the argument we can show that the symmetrically balanced imcomplete block design can by interchanging the constituents within the blocks, be converted into a rectangular arrangement in which every row will contain the treatments $1, 2, \ldots, v$ in some order. Such an arrangement will be called a Youden Square.

(*i*) Analysis without recovery of interblock information.

In the notation of Section 2 of Chapter 2

$$\mathbf{L} = \mathbf{E}_{vr},\ \mathbf{M} = \mathbf{N},\ \mathbf{Q} = \mathbf{T} - \frac{\mathbf{NB}}{r}$$

$\mathbf{F} = \dfrac{\lambda v}{r}\ \mathbf{I} - \dfrac{\lambda}{r}\ \mathbf{E}_{vv}$. Hence a solution of (2.2.7) is

$$\hat{\boldsymbol{\tau}} = \frac{r}{\lambda v}\ \mathbf{Q} \tag{3.6.1}$$

Comparison with (3.5.12) shows that the analysis is exactly similar to the analysis of the corresponding symmetrical B I B D ; only the sum of squares due to replications (rows) is separated out from the error sum of squares in the Analysis of Variance Table of the corresponding B I B D. After the experiment is over, one may be interested in knowing whether this additional feature had been worthwhile or not. An estimate of the per plot variance of the corresponding B I B D can be obtained from

$$\frac{\text{Sum of Squares due to replications} + (v-1)(r-1)E_Y}{v(r-1)} \tag{2.6.2}$$

where $E_Y$ is the error mean square in the Analysis of Variance Table for the Youden Square. If $E_Y$ is substantially less than (3.6.2), the Youden Square arrangement will lead to increased precision. In comparing the relative magnitudes of these two estimates of per plot variance, the fact that the error sum of squares in the Youden Square is based on a lesser number of degrees of freedom should be taken into account.

(*ii*) Analysis with recovery of interblock information.

Denoting by $y_{ij}$, the yield of the plot in the $j$ th column of the $i$ th row, we have

$$E\,|\,(y_{ij}\,|\,\beta_1,\beta_2,\ldots,\beta_v)=\mu+\rho_i+\beta_j+\tau_{(ij)}$$
$$V\,|\,(y_{ij}\,|\,\beta_1,\beta_2,\ldots,\beta_v)=\sigma_1{}^2$$

where $\rho_i$ is the effect of $i$ th row (replication), $\tau_{(ij)}$ the effect of the treatment applied to $(ij)$th plot and $\beta_j$'s are block (column) effects which are independent normal with mean zero and variance $\sigma_2^2$: Hence

$$E(y_{ij})=EE(y_{ij}\,|\,\beta_1,\ldots,\beta_v)=\mu+\rho_i+\tau_{(ij)}$$
$$V(y_{ij})=E\,V(y_{ij}\,|\,\beta_1,\ldots,\beta_v)+v[E(y_{ij}\,|\,\beta_1,\ldots,\beta_v)]$$
$$=\sigma_1^2+\sigma_2^2$$

$$\mathrm{Cov}(y_{ij},y_{i'j'})=E\,\mathrm{Cov}(y_{ij},y_{i'j'}\,|\,\beta_1,\ldots,\beta_v)+\mathrm{Cov}[E(y_{ij}\,|\,\beta_1,\ldots,\beta_v),\;E(y_{i'j'}\,|\,\beta_1,\ldots,\beta_v)]$$
$$=\sigma_2^2\text{ if }i=i',\;j\neq j'\text{ and }0\text{ otherwise}$$

The analysis now follows as in section 2 of chapter 2

From ( 3.5.18 ) and Table we have

$$\text{Est }\nu=\frac{E_b-E_a}{v\,(r-1)E_b}$$

The best unbiased linear estimate of $\mu+\tau_s$ is

$$\frac{1}{r}\,(T_s+\nu\,W_s)$$

The variance of the best estimate of $\tau_s-\tau_{s'}$ is estimated by

$$\frac{2\,E_e}{r}\,[\,1+(v-k)\,\nu\,]$$

7. *Lattice Design.* Let us have $k^2$ treatments arranged in $k$ rows and $k$ columns. Then if we assign the treatments occurring in rows and columns to different blocks, we have $2k$ blocks of $k$ plots each. Again if $m-2$, $(m-2\leqslant k-1)$ mutually orthogonal Latin Squares exist, we can superimpose these Latin Squares on the $k^2$ treatments arranged in $k$ rows and $k$ columns and assign to blocks treatments corresponding to letters of these squares. Such an arrangement will be called an *m-ple* Lattice. We shall further suppose that the whole arrangement is replicated $n$ times. Thus $v=k^2$, $b=nmk$, $r=mn$ and in $C=(c_{ss'})$, $c_{ss}=r\left[1-\dfrac{1}{k}\right]$ and $c_{ss'}$ when $s\neq s'$ is either $n/k$ or $o$ according as the treatments $s$ and $s'$ occur together in a block or not. Let $S_1(\tau_s)$, $S_2(\tau_s)$, $S_i(\tau_s)$ denote respectively the sums of the effects of the treatments (including the $s$ th) occurring in a row, in a

TABLE 3.6.1

*Analysis of Variance Table for Youden Square*

| Source | DF | SS | MEAN SS | E(SS) |
|---|---|---|---|---|
| Replications (rows) | $r-1$ | $\frac{1}{v}\sum_{i=1}^{r} R_i^2 - \frac{G^2}{vr}$ | ... | $(r-1)\,\sigma_1^2 + v\sum(\rho_i - \bar{\rho})^2$ |
| Blocks (eliminating treatments) | $v-1$ | $\sum_{s=1}^{v} W_s^2/v\lambda\,(v-1)(v-k)$ | $E_b$ | $(v-1)\,\sigma_1^2 + v\,(r-1)\,\sigma_2^2$ |
| Treatments (unadjusted) | $v-1$ | $\frac{1}{r}\sum_{s=1}^{v} T_s^2 - \frac{G^2}{vr}$ | ... | $(v-1)\,\sigma_1^2 + (v-r)\,\sigma_2^2 + r\sum(\tau_s - \bar{\tau})$ |
| Error (intrablock) | $(v-1)(r-2)$ | — | $E_e$ | $(v-1)(r-2)\,\sigma_1^2$ |
| Total | $vr-1$ | $\sum_{i=1}^{r}\sum_{j=1}^{v} y_{ij}^2 - \frac{G^2}{vr}$ | | $(vr-1)\,\sigma_1^2 + r\,(v-1)\,\sigma_2^2 + v\sum(\rho_i - \bar{\rho})^2 + r\sum(\tau_s - \bar{\tau})^2$ |

column and having the same letter in the $(i-2)$ th Latin Square as the $s$ th, $i = 3, 4, \ldots, m$. The same excluding $\tau_s$ will be represented by $S'_1(\tau_s)$. $S'_2(\tau_s)$ and $S'_i(\tau_s)$. $i = 3, 4, \ldots, m$.

Then the normal equations utilising intrablock information only are

$$Q_s = r\left[1 - \frac{1}{k}\right]\hat{\tau}_s - (n/k)\sum_{i=1}^{m} S'_i(\hat{\tau}_s) \tag{3.7.1}$$

$$= r\hat{\tau}_s - (n/k)\sum_{i=1} S_i(\hat{\tau}_s) \tag{3.7.2}$$

$$\therefore\ S_i(Q_s) = n(m-1)\,S_i(\hat{\tau}_s) \tag{3.7.3}$$

From (3.7.1) and (3.7.2) we have

$$\hat{\tau}_s = \left[Q_s + \sum_{i=1}^{m} S_i(Q_s)/k(m-1)\right]/r \tag{3.7.4}$$

Thus $V(\hat{\tau}_s - \hat{\tau}_{s'}) = 2\sigma^2(k+1)/kr$ or $2\sigma^2[k+m/m-1]/kr$ according as treatments $s$ and $s'$ occur or do not occur together in a block.

The following normal equations combining inter and intrablock information can be obtained from (3.7.1) by following the method of Ex. 17 of chapter 2.

$$P_s = r[w + w'/(k-1)]\,\frac{k-1}{k}\,\hat{\tau}_s - \frac{(w-w')n}{k}\sum_{i=1}^{m} S_i(\hat{\tau}_s) \tag{3.7.5}$$

$$= rw\,\hat{\tau}_s - (w-w')n\sum_{i=1}^{m} S_i(\hat{\tau}_s)/k \tag{3.7.6}$$

whence $S_i(P_s) = n(\overline{m-1}\,w + w')\,S_i(\hat{\tau}_s)$ (3.7.7)

$$\therefore\ \hat{\tau}_s = \left[P_s + (w-w')\sum_{i=1}^{m} S_i(P_s)/k(\overline{m-1}\,w + w')\right]/rw \tag{3.7.8}$$

From (3.7.8), $V(\hat{\tau}_s - \hat{\tau}_{s'})$ is

$(2/rw)\,\{1 + (m-1)(w-w')/k(\overline{m-1}\,w + w')\}$ or

$(2/rw)\,\{1 + m(w-w')/k(\overline{m-1})\,w + w')\}$ according as the treatments $s$ and $s'$ do or do not occur together in a block.

8. *Partially Balanced Incomplete Block Designs* (PBIBD). An incomplete block design is said to be partially balanced when

(*i*) There are $v$ treatments arranged in $b$ blocks each containing $k$ plots with different treatments assigned to each;

(*ii*) each treatment occurs in $r$ blocks ;

(*iii*) with respect to any treatment the remaining can be divided into $m$ groups containing $n_1, n_2, \ldots, n_m$ treatments, such that the treatments of the $i$ th group occurs with the given treatment $\lambda_i$ times. The treatments of the $i$ th group are said to be $i$-associates of the given treatment. The numbers $n_1, n_2, \ldots, n_m$ ; $\lambda_1, \lambda_2 \ldots, \lambda_m$ are independent of the treatment with which we start. Some of the $\lambda$'s may be equal.

(*iv*) if the treatment $A$ is $i$-associate of $B$, then treatment $B$ is $i$-associate of $A$. If $A$ and $B$ are $i$-associates, then the number of treatments common to the $j$-associates of $A$ and $j'$-associates of $B$ is $p^i_{jj'}$ and is independent of the pair of treatments we start with. Also $p^i_{jj'} = p^i_{j'j}$.

Let N be the incidence matrix of the design. Then (3.5.1.) is valid and we have

$$kb = vt \tag{3.8.1}$$

From condition (*iii*), we have

$$n_1 + n_2 + \ldots + n_m = v - 1 \tag{3.8.2}$$

The matrix N N′ consists of $r$ in the principal diagonal and $n_1$ times $\lambda_1$, $n_2$ times $\lambda_2, \ldots, n_m$ times $\lambda_m$ in each row and column. Thus

$$\mathbf{NN'}\,\mathbf{E}_{v1} = \left[ r + \sum_{i=1}^{m} n_i \lambda_i \right] \mathbf{E}_{v1} = \mathbf{N}\,(\,\mathbf{N'E}_{v1}\,) = (\,rk)\,\mathbf{E}_{v1}$$

$$\therefore \sum_{i=1}^{m} n_i \lambda_i = r\,(\,k - 1\,) \tag{3.8.3}$$

The parameters $b, v, r, k, \lambda_1\, \lambda_2, \ldots, \lambda_m, n_1, n_2, \ldots, n_m$ are known as primary parameters. From (3.8.1), (3.8.2) and (3.8.3), it is evident that there are $2m + 1$ independent primary parameters.

$$p^i_{jj'} = p^i_{j'j} \tag{3.8.4}$$

Let $A$ and $B$ be two $i$-associates. Then $B$ is contained in the $n_i$ variates which are $i$-associates of $A$. Among the other $n_i - 1$ $i$-associates of $A$, there are exactly $p^i_{ij}$ varieties which are $j$-associates of $B$.

Therefore, $\sum\limits_{j=1}^{m} p^i_{ij} = n_j - 1$. If $i \neq j$, among the $n_j$ varieties which are $j$-associates of $A$, there are exactly $p^i_{jj'}$ varieties which are at the same

time $j'$-associates of $B$. Thus $\sum_{j=1}^{m} p^i_{jj'} = n_i$. Thus

$$\sum_{i=1}^{m} p^i_{jj'} = n_j - 1 \text{ if } j = i$$

$$\text{and } = n_j \qquad \text{if } j \neq i \tag{3.8.5}$$

Consider the group consisting of the $i$ th associates of a variety and the group consisting of the $j$ th associates of the same variety. The number of varieties common between $j'$-associates of variety of the 1st group and the varieties of the second group is $p^i_{jj'}$. Hence the number of $j'$-associates that can be formed by taking a variety of the group and a variety of the second group is on the one hand $n_i\, p^i_{jj'}$ and on the other $n_j\, p^j_{ij'}$.

Hence

$$n_i p^i_{jj'} = n_j p^j_{ij'} \tag{3.8.6}$$

Thus the $m$ symmetric matrices

$$\mathbf{P}_1 = [p^1_{jj'}]\,,\ \mathbf{P}_2 = [p^2_{jj'}]\,,\ \ldots\ \mathbf{P}_m = [p^m_{jj'}] \tag{3.8.7}$$

contain only $\dfrac{m\,(m^2-1)}{6}$ independent elements. Consider the $j'$-associate classes of the $n_i$ $i$-associates of a treatment $A$. How many times will a particular treatment $B$ belonging to the $j$-associate class of $A$ occur in the above set? How many times will $A$ itself occur? Let $B$ occur $x$ times and for the sake of definiteness, let it occur in the $j'$-associate classes of $A_{i_1}$ $A_{i_2}, \ldots, A_{i_x}$. Then $A_{i_1}\ A_{i_2} \ldots, A_{i_x}$ will occur in the $j'$-associate class of $B$ and hence $x =$ number of common varieties between the $j'$-associate class of $B$ and the $i$-associate class of $A$ i.e. $p^{\,i}_{ij'}$. In the $j'$-associate classes of $A_{i_1}, A_{i_2} \ldots, A_{i_{n_i}}$, $A$ cannot occur unless $j' = i$, in which case it will occur $n_i$ times. Let $S_i(\tau_s)$ denote summation over the $i$-associates of $\tau_s$. Then the above result is equivalent to

$$S_{j'} = S_i(\tau_s) = \sum_{j=1}^{m} p^j_{ij'}\, S_j(\tau_s) \text{ if } j' \neq i$$

and
$$= n_i \tau_s + \sum_{j=1}^{m} p^j_{ii}\, S_j(\tau_s) \quad \text{if } j' = i. \tag{3.8.8}$$

(*i*) Analysis without recovery of interblock information. Equation (2.1.29) can be written in this case as

$$Q_s = r\left(1 - \frac{1}{k}\right)\hat{\tau}_s - \frac{1}{k}\sum_{i=1}^{m} \lambda_i S_i(\hat{\tau}_s)$$

$$\therefore\ S_{j'}(Q_s) = r\left(1 - \frac{1}{k}\right) S_{j'}(\hat{\tau}_s) - \frac{1}{k}\sum_{i=1}^{m} \lambda_i S_{j'} S_i(\hat{\tau}_s) \tag{3.8.9}$$

Remembering (3.7.8), and imposing the condition $\hat{\tau}_s + S_1(\hat{\tau}_s) + \dots + S_m(\hat{\tau}_s) = 0$

$$k\,S_i(Q_s) = a_{i1} S_1(\hat{\tau}_s) + a_{i2} S_2(\hat{\tau}_s) + \dots + a_{im} S_m(\hat{\tau}_s)$$

$$i = 1, 2, \dots, m \tag{3.8.10}$$

where $a_{ij} = \lambda_i n_i - \sum_{l=1}^{m} \lambda_l p_{il}^{j}\ ,\ i \neq j$

$$a_{ii} = r(k-1) + \lambda_i n_i - \sum_{l=1}^{m} \lambda_l p_{il}^{i} \quad i, j = 1, 2, \dots, m. \tag{3.8.11}$$

From (3.8.10), we can write

$$S_i(\hat{\tau}_s) = f_{i1} S_1(Q_s) + f_{i2} S_2(Q_s) + \dots, + f_{im} S_m(Q_s)$$

$$i = 1, 1, \dots, m. \tag{3.8.12}$$

and finally

$$r(k-1)\tau_s = k Q_s + \sum_{j=1}^{m} S_j(Q_s) \sum_{i=1}^{m} \lambda_i f_{ij} \tag{3.7.13}$$

Hence if the $\tau_s - \tau_{s'}$ is estimable and if $s$ and $s'$-th treatments are $i$-associates, the best estimate of $\tau_s - \tau_{s'}$ is

$$\hat{\tau}_s - \hat{\tau}_{s'} \tag{3.8.14}$$

with variance

$$\frac{2\sigma^2}{r(k-1)}\left[k - \sum_{l=1}^{m} \lambda_l f_{li}\right] \tag{3.8.15}$$

(*ii*) Analysis with recovery of interblock information.

Following the procedure in Ex. 17 chapter 2 we have

$$P_s = w Q_s + w' Q'_s = r\left[w + \frac{w'}{k-1}\right]\frac{k-1}{k}\hat{\tau}_s - \frac{w - w'}{k}\sum_{l=1}^{m} \lambda_l S_l(\hat{\tau}_s)$$

$$\therefore \quad S_i(P_s) = r\left[w + \frac{w'}{k-1}\right]\frac{k-1}{k}\; S_i(\hat{\tau}_s) - \frac{w-w'}{k}\sum_{l=1}^{m}\lambda_l\, S_i\, S_l(\hat{\tau}_s)$$

$$= b_{i1}\, S_1(\hat{\tau}_s) + b_{i2}\, S_2(\hat{\tau}_s) + \dots + b_{im}\, S_m(\hat{\tau}_s)$$

where $b_{ii} = \frac{1}{k}\left[r\left\{w(k-1) + w'\right\} + n_i\lambda_i(w-w') - (w-w')\sum_{l=1}^{m}\lambda_l\, p_{li}^{i}\right]$

$$b_{ij} = \frac{w-w'}{k}\left[n_i\lambda_i - \sum_{l=1}^{m}\lambda_l\, p_{li}^{j}\right],\ i \neq j$$

whence $S_i(\hat{\tau}_s) = \sum_{j=1}^{m} g_{ij}\, S_j(P_s)$

$$i = 1, 2, \dots, m$$

$$\therefore \quad \hat{\tau}_s = \frac{k\,P_s}{r\,[w(k-1)+w']} + \frac{w-w'}{r\,[w(k-1)+w']}\sum_{l=1}^{m}\lambda_l\sum_{j=1}^{m} g_{lj}\, S_j(P_s)$$

Hence if the $s$ and $s'$-th treatments are $i$-associates, the variance of the difference of their best estimates is

$$\frac{2}{r\,[w(k-1)+w']}\left\{k - (w-w')\sum_{l=1}^{m}\lambda_l\, g_{li}\right\}$$

## Exercises to Chapter III

1. In a randomised block experiment with $v$ varieties and $r$ replications, variety 2 in Block 1 is interchanged with variety 1 in Block 2. Indicate the structure of analysis of the modified design and give estimates and variances of different treatment comparisons. What is the loss in efficiency due to this interchange?

2. In a randomised block experiment originally planned with $v$ varieties and $r$ replications, it is found later on that there is not enough material of variety 1 and excess of material of variety 2. Variety 2 is then applied twice to blocks 1, 2, ..., $r_1$ and once to the remaining blocks. Variety 1 is applied only in blocks $(r_1+1)$, $(r_1+2)$, ..., $r$. Write down the $C$ matrix of the design and obtain variances of different treatment comparisons. Indicate how you will carry out the analysis of variance.

3. In a randomised block experiment, variety 1 was used twice by mistake in the first two blocks and as a result variety 2 did not occur in these two

blocks. How will you analyse such an experiment and test the overall differences among varieties and also the difference between individual pairs?

4. In an $s \times s$ Latin square, two treatments in one column get interchanged by accident. Obtain estimates of different treatment comparisons and their variances.

5. There are $v$ new varieties to be tested. An experimenter divides his field into $v$ blocks and each block into $tv + 1$ plots. Each new variety is replicated $t$ times in each block and a standard variety is assigned to the $(tv + 1)$ th plot of the block. Give the analysis appropriate to this design and compare it with the arrangement in $tv + 1$ randomised blocks of $v$ plots each.

6. Let $\mathbf{N}$ be the incidence matrix of a symmetrical BIBD. Consider the matrix

$$\mathbf{M} = \begin{bmatrix} -k\mathbf{I}_1 & \sqrt{-\lambda}\ \mathbf{E}_{1v} \\ \sqrt{-\lambda}\ \mathbf{E}_{v1} & \mathbf{N} \end{bmatrix}$$

Show that $\mathbf{MM}' = \mathbf{M}'\mathbf{M} = (r - \lambda)\ \mathbf{I}_{v+1}$ and hence $\mathbf{NN}' = \mathbf{N}'\mathbf{N}$.

7. Let $\mathbf{N}$ be the incidence matrix of a BIBD.

(*i*) Show that det $\mathbf{N}'\mathbf{N} = 0$ when the BIBD is non-symmetrical

(*ii*) Show that the eigen values of $\mathbf{NN}'$ are $rk$ and $r - \lambda$ with multiplicities 1 and $v - 1$ respectively

(*iii*) Obtain a non-singular $\mathbf{G}$ such that

$$\mathbf{G}'\ (\mathbf{NN}')\mathbf{G} = \text{diag}\ (rk,\ r - \lambda,\ r - \lambda, \ldots,\ r - \lambda).$$

8. Denoting by $\mathbf{N}$ the incidence matrix of a BIBD with parameters $v, b, r, k$ and $\lambda$ show that (*i*) $\mathbf{E}_{1b}\ \mathbf{N}'\mathbf{N} = rk\ \mathbf{E}_{1b}$ (*ii*) $(\mathbf{N}'\mathbf{N})\ (\mathbf{N}'\mathbf{N}) = (r - \lambda)\ \mathbf{N}'\mathbf{N} + k^2\lambda\ \mathbf{E}_{bb}$. Use these results to prove that if a BIBD can be constructed with $v = \frac{1}{2}k(k + 1)$, $b = \frac{1}{2}(k + 1)\ (k + 2)$, $k$, $r = k + 2$, $\lambda = 2$, then between any two blocks there are either 1 or 2 varieties in common. Show further that there are $2\ k$ blocks having 1 variety in common with a given block and $k(k - 1)/2$ blocks having 2 varieties in common with a given block.

9. If $\mathbf{N}_1$ is obtained from $\mathbf{N}$, the incidence matrix of a BIBD by replacing each zero by $-1$, show that $\mathbf{N}_1\ \mathbf{N}'_1 = 4(r - \lambda)\mathbf{I}_v + [b - 4\,(r - \lambda)]\ \mathbf{E}_{vv}$.

Hence $\mathbf{N}_1\ \mathbf{N}'_1 = b\ \mathbf{I}_v$ if $b = 4\ (r - \lambda)$ i.e. $2k = v \pm \sqrt{v}$.

If $\mathbf{N}_1$ and $\mathbf{N}_2$ be matrices obtained from the incidence matrices of two such possible BIBD satisfying these relations, show that the kronecker product $\mathbf{N}_1 \times \mathbf{N}_2$ will lead to another BIBD. Obtain the parameters of this new BIBD.

10. Prove that a symmetrical BIBD can always be constructed with parameters $v = b = 2^m - 1$, $r = k = 2^{m-1}$, $\lambda = 2^{m-2}$. Utilise this for the construction of a new BIBD with parameters $v = 2^{m-1} - 1$, $b = 2\ (2^{m-1} - 1)$, $k = 2^{m-2}$, $r = 2^{m-1}$ and $\lambda = 2^{m-2}$.

11. Show that the Lattice Design of Section 7 is a special case of a PBIBD with two associate classes. Obtain its primary and secondary parameters.

12. There are $m\ k$ treatments arranged in $m$ sets of $k$ each and treatments of a set are assigned to a block and there are $r$ replications. Show that the resulting design is a PBIBD with two associate classes. Obtain the parameters of the design and show that the design is not connected. Demonstrate also that by choosing $r$ properly, we can make the number of blocks equal to, less than or greater than $v$, the number of treatments.

13. An incomplete block design is called a Linked Block design if (*i*) each block has the same number of treatments $k$ (*ii*) each treatment occurs in $r$ blocks (*iii*) any two blocks have the same number of treatments $\lambda^*$ in common. Show that by interchanging the roles of blocks and treatments in a BIBD, one can obtain a Linked Block Design. Obtain the matrix **D** given by (2.1.15) for this design and show that that its rank is $(b-1)$. Show further that the eigen values of $D$ are $o$ and $1/r\ [k\ (r-1)-\lambda^*]$ having multiplicities 1 and $(b-1)$ respectively.

14. By taking $v(v-1)/2$ combinations of $v$ treatments taken two at a time we get a BIBD with parameters $b=v(v-1)/2$, $k=2$, $r=v-1$ and $\lambda=1$. Taking blocks for treatments and treatments for blocks in this design, obtain a new design in which any two blocks have a single treatment as a common link. Show that the new design is partially balanced and obtain the parameters of the design.

15. Show that in the case of PBIBD, the eigen values of NN′ are $r\ k$ and the eigen values of A with appropriate multiplicities where A is the matrix

$(a_{ij})$ and $a_{ij}=\sum\limits_{l=1}^{m}\lambda_l\,p^j_{li}-n_i\,\lambda_i,\ j\neq i,$

$a_{ii}=r+\sum\limits_{l=1}^{m}\lambda_l\,p^i_{li}-n_i\lambda_i,\ i,j=1,2,\ldots,m.$ If NN′ has a zero eigen value with multiplicity $u$ show that $b\geqslant v-u$ and when the design is resolvable $b\geqslant v+(r-1)-u$. In the case of PBIBD with two associate classes, determine the values and multiplicities of the eigen values of NN′ in terms of the parameters of the design.

16. Show that in a PBIBD with $m$ associate classes, the determinant $|\,l_{ij}|$

when $k>r$ has the value zero where $l_{ij}=r\,\delta_{ij}+\sum\limits_{k'=1}^{m}\lambda_{k'}\ p^i_{jk'}-\lambda_j\ n_j$ and

$\delta_{ij}=1$ if $i=j$ and 0 if $i\neq j$.

17. Show that the efficiency factor (ratio of the variance of a varietal comparison in a design in ordinary randomised blocks to the average variance of

the particular design occupying the same number of plots and having the same error variance per plot) for a $p \times q$ lattice is $(pq-q)/(pq+p+q-3)$.

18. Obtain the efficiency of the Lattice design in section 7 when analysis is made (*i*) utilising intrablock information only and (*ii*) with recovery of interblock information.

19. $p \times p \times p$ varieties are arranged in a three way table i.e. spatially in the form of a cube and treatments along lines parallel to the edges of the cube are then assigned to blocks. Obtain the variances of different treatment comparisons. Calculate the efficiency of the design relative to the corresponding randomised block design.

20. Obtain three $4 \times 4$ orthogonal Latin Squares to construct (*i*) a BIBD with parameters $v = b = 21$, $r = k = 5$, $\lambda = 1$, (*ii*) a set of 15 orthogonal comparisons from 16 quantities $x_1, x_2, \ldots, x_{16}$ (*iii*) a balanced Lattice design with 16 varieties and (*iv*) a PBIBD with $v = 16$, $b = 16$, $r = 4$, $k = 4$, $\lambda_1 = 1$, $n_1 = 12$, $\lambda_2 = 0$, $n_2 = 3$, $p^1_{11} = 8$.

21. A BIBD with parameters $v, b, r, k, \lambda$ is said to be affine resolvable if the blocks can be separated into $r$ sets each forming a complete replication and further if any two blocks of different sets have the same number of treatments in common. Show that for such a design (*i*) the parameters can be expressed in terms of two integers $n$ and $t$ $(n \geqslant 2,\ t \geqslant 0)$ as $v = nk = n^2[(n-1)t+1]$, $b = nr = n(n^2+nt+1)$, $\lambda = nt+1$ (*ii*) the number of treatments common to any two blocks of different sets is $k^2/v = (n-1)t+1$.

22. An incomplete block design with $v$ treatments each replicated $r$ times in blocks of size $k$ $(< v)$ is said to be group divisible if the treatments can be divided into $c$ groups each of size $d$ so that any two treatments belonging to the same group occur together in $\lambda_1$ blocks whereas those coming from different groups occur together in $\lambda_2$ $(> 0)$ blocks. Show that (*i*) a necessary condition for the existence of such a design is $rk - \lambda_2 v \geqslant 0$ and (*ii*) the following relations hold between the parameters: $v = cd$, $bk = vr$, $r(k-1) = (d-1)\lambda_1 + d(c-1)\lambda_2$, $r \geqslant \lambda_1$, $r \geqslant \lambda_2$.

23. Denote the design of Ex. 21 by $A(n, t)$ and the following Group Divisible Design by $G(n, t)$:

$$v = b = (n-1)(n^2t+n+1),\ r = k = n[(n-1)t+1].$$
$$\lambda_1 = 0,\ \lambda_2 = (n-1)t+1 \text{ where } c = n^2t+n+1$$
$$\text{and } d = n-1,\ n \text{ and } t \text{ being integers, } n > 2,\ t \geqslant 0.$$

Show that $A(n, t)$ and $G(n,t)$ either both exist or both do not exist.

24. Defining Efficiency as in Ex. 17, obtain an expression for it in the case of an incomplete block design and obtain a criterion so that an incomplete block design may have maximum efficiency.

Establish that in the case of incomplete block designs having an equal number of plots per block, the BIBD is most efficient and in the case of two-way classified incomplete blocks of equal block size, the Youden Square is the most efficient.

25. To each block of a BIBD with parameters $v$, $b$, $r$, $k$ and $\lambda$ a control treatment is added to each block, so that the block size is now $(k+1)$. Give the analysis for the design, estimates of different varietal comparisons and their variances.

CHAPTER IV

# STANDARD DESIGNS—*Continued.*

1. *Factorial Designs.* Often instead of studying factors singly, it is more advantageous to study several factors in all combinations. Let the treatments consist of combinations of factors $F_1, F_2, \ldots, F_m$, the $i$th factor being available at 0th, 1st, 2nd, ..., $(s_i - 1)$th levels. Thus a typical treatment combination in which the $i$th factor is at $x_i$th level $(i = 1, 2, \ldots, m)$ may be denoted by $f_1^{x_1} f_2^{x_2} \ldots f_m^{x_m}$ or by $\tau\,(x_1, x_2, \ldots, x_m)$. A linear function of the treatments

$$\sum C\,(x_1, x_2, \ldots, x_m)\,\tau\,(x_1, x_2, \ldots, x_m) \tag{4·1·1}$$

will be called a contrast if all c's are not zero and

$$\sum C\,(x_1, x_2, \ldots, x_n) = 0 \tag{4·1·2}$$

the summation extending over all values of $x_1, x_2, \ldots, x_m$.

A treatment contrast will be said to belong to the $(k-1)$ th order interaction between the factors $F_{i_1}, F_{i_2}, \ldots, F_{i_k}$ if $(i)$ $C(x_1, x_2, \ldots, x_m)$ depends only on $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ and $(ii)$ the sum of $C\,(x_1, x_2, \ldots, x_m)$ over any one of the arguments $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ is zero. The 0-th order interactions are called Main Effects. If we write $\phi_i = f_i^0 + f_i^1 + \ldots, f^{s_i - 1}$, $i = 1, 2, \ldots, m$, then a typical linear function of treatments belonging to $(k-1)$=th order interaction between the factors $F_{i_1}, F_{i_2}, \ldots, F_{i_k}$ is

$$\frac{\phi_1\,\phi_2 \ldots \phi}{\phi_{i_1}\,\phi_{i_2} \ldots \phi_{i_k}} \sum l\,(x_{i_1}, x_{i_2}, \ldots, x_{i_k})\, f_{i_1}^{x_{i_1}}\, f_{i_2}^{x_{i_2}} \ldots f_{i_k}^{x_{i_k}} \tag{4·1·3}$$

$$\text{where } 0 = \sum_{x_{i_1}} l\,(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) = \sum_{x_{i_2}} l\,(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) = \ldots$$

$$= \sum_{x_{i_k}} l\,(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) \tag{4·1·4}$$

One obvious way of obtaining these linear functions is to choose

$l\,(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) = l_1\,x_{i_1}.\; l_2\,x_{i_2}, \ldots\; l_k\,x_{i_k}$

where $l_{j\,x_i}$ is such that $\sum_{x_{i_j}} l_{j\,x_{i_j}} = 0$

$j = 1, 2, \ldots, k$. Thus there are $(s_{i_1} - 1)(s_{i_2} - 1) \ldots (s_{i_k} - 1)$ independent contrasts

$$\frac{\phi_1 \phi_2 \cdots \phi_m}{\phi_{i_1} \phi_{i_2} \phi_{i_k}} \sum_{x_{i_1}} l_{1\, x_{i_1}} f_{i_1}^{x_{i_1}} \sum_{x_{i_2}} l_{2\, x_{i_2}} f_{i_2}^{x_{i_2}} \cdots \sum_{x_{i_k}} l_{k\, x_{i_k}} f_{i_k}^{x_{i_k}} \qquad (4\cdot1\cdot5)$$

belonging to the $(k - 1) =$ th order interaction between $F_{i_1}, F_{i_2}, \ldots, F_{i_k}$. From the definition, it is evident that (*i*) interactions of different orders are mutually orthogonal (*ii*) two interactions of the same order when the interacting factors differ at least in one element, are mutually orthogonal. Thus the total number of independent treatment contrasts $s_1. s_2. \ldots\ldots s_m - 1$ can be broken up into (1) $s_1 - 1, s_2 - 1, \ldots\ldots, s_m - 1$ sets of independent contrasts belonging to Main Effects (2) $(s_1 - 1)(s_2 - 1), (s_1 - 1)(s_3 - 1), \ldots, (s_{m-1} - 1)(s_m - 1)$ sets of independent contrasts belonging to 1st order interactions, ......($m$) $(s_1 - 1)(s_2 - 1) \ldots\ldots (s_m - 1)$ independent contrasts belonging to $(m - 1)$ —th order interaction. Note also that $s_1 s_2 \ldots\ldots s_m - 1$ $= [1 + (s_1 - 1)]\,[1 + (s_2 - 1)] \ldots [1 + (s_m - 1)] - 1$

$$= \sum_{i=1}^{m} (s_i - 1) + \sum_{i \neq j} (s_j - 1\,(s_i - 1) + \ldots + (s_1 - 1)(s_2 - 1) \ldots (s_m - 1).$$

Let the factorial experiment be carried out in $r$ randomised blocks of $(s_1 s_2 \ldots s_m)$ plots each. Then the best estimate of a treatment contrast is the corresponding linear function of the mean yields. Let $T_{i_1 i_2 \ldots\ldots i_k}$ $(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$ denote the total of treatments in which the $i_1$ th factor is at $x_{i_1} =$ th level, $i_2$ th factor is at the $x_{i_2}$ —th level, ... and $i_k$ the factor is at $x_{ik}$ — th level. Then the sum of squares due to main effects of $F_1$ is

$$\sum_{x_1=0}^{s_1-1} \frac{T_1^2(x_1)}{r\, s_2\, s_3 \ldots s_m} - \frac{T^2}{r\, s_1\, s_2 \ldots s_m}, \; T \text{ denoting}$$

the grand total. The sum of squares due to interaction between $F_1$ and $F_2$ is

$$\sum_{x_1, x_2} \frac{T_{12}^2(x_1, x_2)}{r\, s_3\, s_4 \ldots s_m} - \text{Sum of squares due to}$$

main effects of $F_1$ — Sum of squares due to main effects of $F_2$

$$+ \frac{T^2}{r\, s_1\, s_2 \ldots s_m}$$

The calculation of sum of squares due to higher order interactions is also obvious. When $s_1 = s_2 = \ldots = s_m = s$( say ), the factorial design is said to be symmetrical and often it will be expressed by $s^m$.

2. *Split-Plot Experiments.* Here for experimental convenience, the plots of a block containing a main treatment are split up into a number of sub-plots to accommodate an equal number of subplot treatments. The subplot yields on account of having the same main treatment common and on account of the contiguity of the plots, are expected to be correlated. Let $y_{ijk}$ denote the yield of the plot containing the $k$ th sub treatment of the $j$ th main treatment in the $i$ th block ( $i = 1, 2, \ldots, r$; $j = 1, 2, \ldots \alpha$ ; $k = 1, 2, \ldots \beta$) Then

$$y_{ijk} = \mu + \alpha_i + \rho_j + \tau_k + \delta_{jk} + \varepsilon_{ijk} \tag{4.2.1}$$

Where $\alpha_i$ denotes the effect of the $i$ th block, $\rho_j$, the effect of $j$ th main treatment, $\tau_k$ the effect of $k$ th subtreatment and $\delta_{jk}$, the interaction between $j$ th main treatment and $k$ th subtreatment and $\varepsilon_{ijk}$' s are normal variates.

$$E\,(E_{ijk}) = 0$$

$$V(\varepsilon_{ijk}) = V(y_{ijk}) \qquad = \sigma^2$$

Cov $(y_{ijk}, y_{i'j'k'}) = \rho\,\sigma^2$ if $i = i'$, $j = j'$, $k \neq k'$, 0 otherwise. We also assume

$$\sum_i \alpha_i = \sum_j \rho_j = \sum_k \tau_k = \sum_j \delta_{jk} = \sum_k \delta_{jk} = 0 \tag{4.2.2}$$

Let

$$\begin{bmatrix} y_{ij1} \\ y_{ij2} \\ \cdots \\ y_{ij\beta} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{\beta}} & b_{12} & b_{11} & \cdots & b_1\beta-1 \\ \frac{1}{\sqrt{\beta}} & b_{22} & b_{21} & \cdots & b_2\beta-1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\sqrt{\beta}} & b_{\beta 1} & b_{\beta 2} & \cdots & b_{\beta\beta}-1 \end{bmatrix} \begin{bmatrix} U_{ij} \\ Z_{ij1} \\ \cdots \\ Z_{ij\,(\beta-1)} \end{bmatrix} \tag{4.2.3}$$

$$i = 1, 2, \ldots, r; \; j = 1, 2, \ldots, \alpha.$$

where the first matrix on the right hand side of (4.2.3) is orthogonal. It is easy to verify

$$V(U_{ij}) = \sigma_2^2 = \sigma^2 \left[ 1 + (\beta - 1)\,\rho \right] = \frac{1}{w'}\,, \quad V(Z_{ij}\lambda) = \sigma_2^2 = \sigma^2\,(1 - \rho) = \frac{1}{w}$$

and all covariances are zero ; $i = 1, 2, \ldots, r$ ; $j = 1, 2, \ldots \alpha$,

$\lambda = 1, 2, \ldots, \beta - 1$. The normal equations are obtained by equating the differential coefficients of

$$w \sum_{i=1}^{r} \sum_{j=1}^{\alpha} \sum_{\lambda=1}^{\beta-1} \Big[ Z_{ij\lambda} - E(Z_{ij\lambda}) \Big]^2 + w' \sum_{i=1}^{r} \sum_{j=1}^{\alpha} \Big[ U_{ij} - E(U_{ij}) \Big]^2$$

$$= w \sum_{i=1}^{r} \sum_{j=1}^{\alpha} \sum_{\lambda=1}^{\beta-1} \Big[ Z_{ij\lambda} - \sum_{k=1}^{\beta} b_{k\lambda} ( \tau_k + \delta_{jk} ) \Big]^2 +$$

$$w' \sum_{i=1}^{r} \sum_{j=1}^{\alpha} \Big\{ U_{ij} - [ \sqrt{\beta} ( \mu + \alpha_i + \rho_j ) ] \Big\}^2$$

with respect to the parameters to zero. They are

$$w' r \alpha \beta y_{\ldots} = w' [ r \alpha \beta \hat{\mu} ]$$

$$w' \alpha \beta y_{i..} = w' [ \alpha \beta \hat{\mu} + \alpha \beta \hat{\alpha}_i ]$$

$$i = 1, 2, \ldots, r$$

$$w' r \beta y_{\cdot j \cdot} = w' [ r \beta \hat{\mu} + r \beta \hat{\gamma}_j ]$$

$$j = 1, 2, \ldots \alpha$$

$$w [ r \alpha ( y_{\cdot\cdot k} - y_{\ldots} ) ]$$

$$+ w' [ r \alpha y_{\ldots} ] = w ( r \alpha \hat{\tau}_k ) + w' r \alpha \hat{\mu}$$

$$k = 1, 2, \ldots, \beta$$

$$w [ r ( y_{\cdot jk} - y_{\cdot j \cdot} ) + w' [ r y_{\cdot j \cdot} ] = w r ( \tau_k + \hat{\delta}_{jk} ) + w' r ( \hat{\mu} + \hat{\gamma}_j )$$

$$j = 1, 2, \ldots, \alpha; \ k = 1, 2, \ldots, \beta. \tag{4.2.4}$$

From ( 4.2.4 ) we easily get the results in Table 4.2.1. below.

TABLE : 4.2.1

*Best Linear Estimates of Some Linear Function of Parameters with Variances*

| Linear Function of Parameters | Best Linear Estimate | Variance of best Linear Estimate |
|---|---|---|
| $\alpha_i - \alpha_{i'}$ | $y_{i..} - y_{i'..}$ | $2\sigma_2^2 / \alpha \beta$ |
| $\rho_j - \rho_{j'}$ | $y_{\cdot j.} - y_{\cdot j'.}$ | $2\sigma_2^2 / r \beta$ |
| $\tau_k - \tau_{k'}$ | $y_{..k} - y_{..k'}$ | $2\sigma_1^2 / r\alpha$ |
| $\delta_{jk} - \delta_{j'k}$ | $y_{\cdot jk} - y_{\cdot j'k} - (y_{\cdot j.} - y_{\cdot j'.})$ | $\frac{2\sigma_1^2}{r} \left( 1 - \frac{1}{\beta} \right)$ |
| $\delta_{jk} - \delta_{jk'}$ | $y_{\cdot jk} - y_{\cdot jk'} - (y_{..k} - y_{..k'})$ | $\frac{2\sigma_1^2}{r} \left( 1 - \frac{1}{\alpha} \right)$ |

$\hat{\mu} = y_{...}$, $\hat{\alpha}_i = y_{i..} - y_{...}$, $\hat{\rho}_j = y_{.j.} - y_{...}$, $\hat{\tau}_k = y_{..k} - y_{...}$ and $\hat{\delta}_{jk} = y_{.jk} - y_{.j.} - y_{..k} + y_{...}$ is a set of solutions of the normal equations (4.2.4). Hence

Regression sum of squares

$$= w' \left[ r\alpha\beta\, y^2_{...} + \alpha\beta \Sigma (y_{i..} - y_{.})^2 + r\beta\Sigma (y_{.j.} - y_{...})^2 + w \left[ r\alpha \Sigma (y_{..k} - y_{...})^2 + r \Sigma\Sigma (y_{.jk} - y_{.j.} - y_{..k} + y_{...})^2 \right]\right.$$

Error sum of squares

$$= w' \beta \Sigma\Sigma (y_{ij\cdot} - y_{i..} - y_{.j.} + y_{...})^2 + w\Sigma\Sigma\Sigma (y_{ijk} - y_{\cdot ij.} - y_{\cdot jk} + y_{\cdot j\cdot})^2$$

Optimum tests are not available if $w'/w$ or $\rho$ is unknown. In the absence of knowledge of $\rho$, we can, however, perform certain exact tests of significance. For if we set

$$\varepsilon_{ijk} = \frac{1}{\sqrt{\beta}} \eta_{ij} + \sum_{\lambda=1}^{\beta-1} b_{k\lambda} \zeta_{ij\lambda} \tag{4.2.5}$$

where the $b$'s are the same as in (4.2.3), we have $V(\zeta_{ij\lambda}) = 1/w$, $V(\eta_{ij}) = 1/w'$, all covariances and expectations are zero.

Now

$$\sum_{i=1}^{r} \sum_{j=1}^{\alpha} \sum_{\lambda=1}^{\beta-1} \zeta^2_{ij\lambda} = r\alpha \sum_{\lambda} \zeta^2_{...\lambda} + r \sum_{j} \sum_{\lambda} (\zeta_{.j\lambda} - \zeta_{...\lambda})^2$$

$$+ \sum_{i} \sum_{j} \sum_{\lambda} (\zeta_{ij\lambda} - \zeta_{.j\lambda})^2 \tag{4·2·6}$$

$$\sum_{i=1}^{r} \sum_{j=1}^{\alpha} \eta_{ij}^2 = r\alpha\eta^2_{..} + r\sum_{j} (\eta_{\cdot j} - \eta_{..})^2 + \Sigma\Sigma (\eta_{ij} - \eta_{i\cdot} - \eta_{\cdot j} + \eta_{..})^2 \tag{4.2.7}$$

By Cochran's rule, the terms on the right hand side of (4.2.6) are distributed independently as $\chi^2 \sigma_1^2$ with $\beta-1$, $(\alpha-1)(\beta-1)$ and $\alpha(r-1)(\beta-1)$ degrees of freedom respectively. Similarly the terms on the right hand side of (4.2.7) are distributed independently as $\chi^2 \sigma_2^2$ with d.f. 1, $\alpha-1$ and $(\alpha-1)(r-1)$ respectively. Now

$$r\Sigma(\eta_{\cdot j} - \eta_{...})^2 = r\beta\Sigma(\varepsilon_{.j\cdot} - \varepsilon_{...})^2 = r\beta\Sigma(y_{.j\cdot} - y_{...})^2, \text{ if}$$
$$\gamma_1 = \gamma_2 = \gamma_3 = \ldots = \gamma_\alpha = 0.$$

$$\alpha\Sigma(\eta_{i\cdot} - \eta_{...})^2 = \alpha\beta\Sigma(\varepsilon_{i\cdots} - \varepsilon_{...})^2 = \alpha\beta\Sigma(y_{i\cdots} - y_{...})^2,$$
$$\text{if } \alpha_1 = \alpha_2 = \ldots = \alpha_r = 0.$$

$$\Sigma\Sigma(\eta_{ij} - \eta_{i\cdot} - \eta_{\cdot j} + \eta_{\cdots})^2 = \beta\Sigma\Sigma(\varepsilon_{ij\cdot} - \varepsilon_{i\cdots} - \varepsilon_{\cdot j\cdot} + \varepsilon_{\cdots})^2$$
$$= \beta\Sigma\Sigma(y_{ij\cdot} - y_{i\cdots} - y_{\cdot j\cdot} + y_{\cdots})^2$$

$$r\alpha \sum_{\lambda=1}^{\beta-1} \zeta^2_{\cdots\lambda} = r\alpha \sum_{k=1}^{\beta} (\varepsilon_{\cdots k} - \varepsilon_{\cdots})^2$$
$$= r\alpha\Sigma(y_{\cdots k} - y_{\cdots})^2, \text{ if } \tau_1 = \tau_2 = \dots \tau_\beta = 0.$$
$$r\Sigma\Sigma\zeta^2_{\cdot i\lambda} = r\Sigma\Sigma(\varepsilon_{\cdot jk} - \varepsilon_{\cdot j\cdot} - \varepsilon_{\cdots k} + \varepsilon_{\cdots})^2$$
$$= r\Sigma\Sigma(y_{\cdot jk} - y_{\cdot j} - y_{\cdots k} + y_{\cdots})^2, \text{ if } \delta_{jk} = 0,\ j = 1, 2, \dots, \alpha;$$
$$k = 1, 2, \dots, \beta.$$
$$\Sigma\Sigma\Sigma(\zeta_{ij\lambda} - \zeta_{\cdot j\lambda})^2 = \Sigma\Sigma\Sigma(\varepsilon_{ijk} - \varepsilon_{ij\cdot} - \varepsilon_{\cdot jk} + \varepsilon_{\cdot j\cdot})^2$$
$$= \Sigma\Sigma\Sigma(y_{ijk} - y_{ij\cdot} - y_{\cdot jk} + y_{\cdot j\cdot})^2$$

Hence we have the following analysis of Variance Table.

TABLE 4.2.2

*Analysis of Variance of Split-Plot Design with Main Treatments, Sub-treatments in r Randomised Blocks.*

| Source | DF | SS |
|---|---|---|
| Blocks | $r-1$ | $\alpha\beta\Sigma(y_{i\cdots} - y_{\cdots})^2$ |
| Main Treatments | $\alpha-1$ | $r\beta\Sigma(y_{\cdot j\cdot} - y_{\cdots})^2$ |
| Error $(a)$: | | |
| Main Treatments × Blocks | $(\alpha-1)(r-1)$ | $\beta\Sigma\Sigma(y_{ij\cdot} - y_{i\cdots} - y_{\cdot j\cdot} + y_{\cdots})^2$ |
| Total | $r\alpha-1$ | $\beta\Sigma\Sigma(y_{ij\cdot} - y_{\cdots})^2$ |
| Subtreatments | $\beta-1$ | $r\alpha\Sigma(y_{\cdots k} - y_{\cdots})^2$ |
| Interaction: | | |
| Main Treatment × Sub-treatment | $(\alpha-1)(\beta-1)$ | $r\Sigma\Sigma(y_{\cdot jk} - y_{\cdot j\cdot} - y_{\cdots k} + y_{\cdots})^2$ |
| Error $(b)$: | $\alpha(r-1)(\beta-1)$ | $\Sigma\Sigma\Sigma(y_{ijk} - y_{ij\cdot} - y_{\cdot jk} + y_{\cdot j\cdot})^2$ |
| Total | $\alpha r(\beta-1)$ | $\Sigma\Sigma\Sigma(y_{ijk} - y_{ij\cdot})^2$ |
| Total | $r\alpha\beta-1$ | $\Sigma\Sigma\Sigma(y_{ijk} - y_{\cdots})^2$ |

For testing Block S S and Main Treatment S S, we have to use Error ($a$) and for testing subtreatments and Interaction between Subtreatments and Main Treatments we have to use Error ($b$).

Let us try to compare the efficiency of the split-plot design with a randomised block design having $\alpha\beta$ plots per block. Writing $\tau_{jk} = \gamma_j + \tau_k + \delta_{jk}$, we can obtain the following table of expectations :

TABLE : 4.2.3.

*Expectations of sums of squares in Split-Plot Design and the Corresponding Expectations had the experiment been designed in r Randomised Blocks of $\alpha\,\beta$ plots each.*

| Split-plot Design | | Randomised Block Design | |
|---|---|---|---|
| Source | E(SS) | Source | E(SS) |
| Main Treatments | $r\beta\Sigma(\tau_{j.}-\tau_{..})^2 + (\alpha-1)\sigma_2^2$ | | |
| Error ($a$) | $(r-1)(\alpha-1)\sigma_2^2$ | | |
| Sub treatments | $r\alpha(\tau_{.k}-\tau_{..})^2 + (\beta-1)\sigma_1^2$ | Treatments | $r\Sigma\Sigma(\tau_{jk}-\tau_{..})^2 + (\alpha\beta-1)\sigma_{RB}^2$ |
| Error ($b$) | $\alpha(r-1)(\beta-1)\sigma_1^2$ | Error | $(\alpha\beta-1)(r-1)\sigma_{RB}^2$ |
| Total | $r\Sigma\Sigma(\tau_{jk}-\tau_{..})^2 + r(\alpha-1)\sigma_2^2 + \alpha r(\beta-1)\sigma_1^2$ | Total | $r\Sigma\Sigma(\tau_{jk}-\tau_{..})^2 + r(\alpha\beta-1)\sigma_{RB}^2$ |

$$\therefore \sigma^2_{RB} = \frac{(\alpha-1)\sigma_2^2 + \alpha(\beta-1)\sigma_1^2}{(\alpha\beta-1)} = \sigma_1^2 + \frac{\alpha-1}{\alpha\beta-1}(\sigma_2^2-\sigma_1^2)$$

So the subtreatments and interactions between main treatment and subtreatment are measured with greater precision whereas the main treatments are measured with less precision and the overall precision is the same in both the cases.

3. *Split-Plot in a Latin Square.* Here the main treatments are arranged in a Latin Square and the plots are further split to accommodate an equal number of subplot treatments. Let $y_{ijk(l)}$ denote the yield of the sub-plot in the $ij$ the plot having $k$ th subtreatment, the main treatment being $l$, $i, j, l = 1, 2, \ldots, t$; $k = 1, 2, \ldots, r$. We can write

$$y_{ijk} = \mu + \alpha_i + \beta_j + \rho_l + \delta_k + \tau_{kl} + \varepsilon_{ijk}$$

$$\text{where } E(\varepsilon_{ijk}) = 0,\ V(\varepsilon_{ijk}) = \sigma^2 \text{ and}$$

$$\text{Cov}\ (\varepsilon_{ijk}, \varepsilon_{i'j'k'}) = \rho\sigma^2 \text{ if } i = i',\ j = j',\ k \neq k' \text{ and zero otherwise.}$$

$$\text{Let } \eta_{ij\lambda} = [\varepsilon_{ij1} + \varepsilon_{ij2} + \ldots + \varepsilon_{ij\lambda} - \varepsilon_{ij(\lambda+1)}]\sqrt{\lambda(\lambda+1)}$$

$$i, j = 1, 2, \ldots, t,\ \lambda = 1, 2, \ldots, r-1$$

$$u_{ij} = \eta_{ijr} = (\varepsilon_{ij1} + \varepsilon_{ij2} \cdots + \varepsilon_{ijr}) / \sqrt{r}\ ,\ i, j = 1, 2, \ldots, t$$

Then expected values of the new variables are zero and covariance between any two of them is zero. Also

$$V(\eta_{ij\lambda}) = \sigma^2(1-\rho) = \sigma_1^2,\ \ V(u_{ij}) = \sigma^2[1 + (r-1)\rho] = \sigma_2^2.$$

Now

$$\text{I} \quad \sum_{i=1}^{t}\sum_{j=1}^{t} u_{ij}^2 = t^2 u^2_{..} + t\sum_{i=1}^{t}(u_{i\cdot} - u_{..})^2 + t\sum_{j=1}^{t}(u_{\cdot j} - u_{..})^2$$
$$+ t\sum_{l=1}^{t}(u_{\cdot(l)} - u_{..})^2 + \sum_{i=1}^{t}\sum_{j=1}^{t}(u_{ij} - u_{i\cdot} - u_{\cdot j} - u_{\cdot(l)} + 2u_{..})^2$$

$$\text{II} \quad \sum_{i=1}^{t}\sum_{j=1}^{t}\sum_{\lambda=1}^{r-1}\eta^2_{ij\lambda} = t^2\sum_{\lambda=1}^{r-1}\eta^2_{..\lambda} + t\sum_{j}\sum_{\lambda}(\eta_{.(l)\lambda} - \eta_{..\lambda})^2$$
$$+ \sum_{i}\sum_{j}\sum_{\lambda}\Big[\eta_{ij\lambda} - \eta_{.(l)\lambda}\Big]^2$$

In I, by Cochran's rule, the constituents on the right hand side are distributed as $\chi^2\sigma_2^2$ with d.f. 1, $t-1$, $t-1$, $t-1$ and $(t-1)$ $(t-2)$ respectively. In II, again by application of Cochran's rule, the constituents on the right hand side are distributed independently as $\chi^2\sigma_1^2$ with d.f. $r-1$, $(t-1)$ $(r-1)$ and $t$ $(t-1)$ $(r-1)$ respectively. Hence we have the following analysis of Variance Table.

TABLE 4.3.1

*Analysis of Variance for Split Plot in $t \times t$ Latin Square*

| Source | D F | S S |
|---|---|---|
| Rows | $t-1$ | $tr \sum_{i=1}^{t} (y_{i\cdot\cdot} - y_{\dots})^2$ |
| Columns | $t-1$ | $tr \sum_{j=1} (y_{\cdot j\cdot} - y_{\dots})^2$ |
| Main Treatments | $t-1$ | $tr \sum_{l=1}^{t} (y_{\cdot\cdot(l)} - y_{\dots})^2$ |
| Error ($a$) | $(t-1)(t-2)$ | $r \sum\sum (y_{ij\cdot} - y_{i\cdot\cdot} - y_{\cdot j\cdot} - y_{\cdot\cdot(l)} + 2y_{\dots})^2$ |
| Total | $t^2-1$ | $r \sum\sum (y_{ij\cdot} - y_{\dots})^2$ |
| Subtreatments | $r-1$ | $t^2 \sum_{k=l}^{r} (y_{\cdot\cdot k} - y_{\dots})^2$ |
| Interaction Main Treatment $\times$ Subtreatment | $(r-1)(t-1)$ | $t \sum\sum (y_{\cdot(l)k} - y_{\cdot\cdot(l)} - y_{\cdot\cdot k} + y_{\dots})^2$ |
| Error ($b$) | $t(t-1)(r-1)$ | $\sum\sum\sum (y_{ijk} - y_{\cdot jk} - y_{ij\cdot} + y_{\cdot j.})^2$ |

4. *Repeated Subdivision.* In section 2, if the subplots are further divided into $\gamma$ sub-sub-plots to accommodate an equal number of sub-sub treatments, our set-up is

$$y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_k + \tau_l + \delta_{jk} + \theta_{kl} + \psi_{jl} + \phi_{jkl} + \varepsilon_{ijkl}$$

with $o = \sum_{i=1}^{r} \alpha_i = \sum_{j=1}^{\alpha} \beta_j = \sum_{k=1}^{\beta} \gamma_k = \sum_{l=1}^{\gamma} \tau_l = \sum_j \delta_{jk} = \sum_k \delta_{jk}$

$= \sum_k \theta_{kl} = \sum_l \theta_{kl} = \sum_j \psi_{jl} = \sum_l \psi_{jl} = \sum_j \phi_{jkl} = \sum_k \phi_{jkl} = \sum_l \phi_{jkl}$

$E(\varepsilon_{ijkl}) = 0$, $V(\varepsilon_{ijkl}) = \sigma^2$

$\text{Cov}(\varepsilon_{ijkl}, \varepsilon_{i'j'k'l'}) = \rho_1 \sigma^2$ if $i = i', j = j', k = k', l \neq l'$

and $= \rho_2 \sigma^2$ if $i = i', j = j', k \neq k'$

and $= 0$ otherwise

Let us consider the three sets of variates

I. $\eta_{ijk\lambda}$, $i = 1, 2, \ldots, r; j = 1, 2, \ldots, \alpha;\ k = 1, 2, \ldots, \beta;\ \lambda = 1, 2, \ldots, \gamma - 1$

II. $\zeta_{ij\nu}$, $i = 1, 2, \ldots, r; j = 1, 2, \ldots, \alpha;\ \nu = 1, 2, \ldots, \beta - 1.$

III. $v_{ij}$, $i = 1, 2, \ldots, r; j = 1, 2, \ldots, \alpha;$

$$\begin{bmatrix} \eta_{ijk1} \\ \eta_{ijk2} \\ \cdots\cdots \\ \eta_{ijk\gamma-1} \end{bmatrix} = \begin{bmatrix} a_{11}\, a_{21} \ldots a\gamma_1 \\ a_{12}\, a_{22} \ldots a\gamma_2 \\ \cdots\cdots\cdots\cdots \\ a_1\gamma_{-1} \ldots a\,\gamma\gamma_{-1} \end{bmatrix} \quad \begin{bmatrix} \varepsilon_{ijk1} \\ \varepsilon_{ijk2} \\ \cdots\cdots \\ \varepsilon_{ijk\gamma} \end{bmatrix} = \mathbf{A} \begin{bmatrix} \varepsilon_{ijk1} \\ \varepsilon_{ijk2} \\ \cdots\cdots \\ \varepsilon_{ijk\gamma} \end{bmatrix}$$

$$\text{where } \begin{bmatrix} \mathbf{A} \\ \mathbf{E}_{1\gamma}/\sqrt{\gamma} \end{bmatrix} \text{ is orthogonal,}$$

$$\begin{bmatrix} \zeta_{ij1} \\ \zeta_{ij2} \\ \cdots\cdots \\ \zeta_{ij\,\beta-1} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{21} & \ldots b_{\beta 1} \\ b_{12} & b_{22} & \ldots b_{\beta 2} \\ \cdots & \cdots & \cdots \\ b_{1\beta-1} & b_{2\,\beta-1} & \ldots b_{\beta\beta-1} \end{bmatrix} \begin{bmatrix} \sqrt{\gamma}\ \varepsilon_{ij1\cdot} \\ \sqrt{\gamma}\ \varepsilon_{ij2\cdot} \\ \cdots\cdots \\ \sqrt{\gamma}\ \varepsilon_{ij\beta\cdot} \end{bmatrix} = \mathbf{B} \begin{bmatrix} \sqrt{\gamma}\ \varepsilon_{ij1\cdot} \\ \sqrt{\gamma}\ \varepsilon_{ij2\cdot} \\ \cdots\cdots \\ \sqrt{\gamma}\ \varepsilon_{ij\beta\cdot} \end{bmatrix}$$

$$\text{where } \begin{bmatrix} \mathbf{B} \\ \mathbf{E}_{1\beta}/\sqrt{\beta} \end{bmatrix} \text{ is orthogonal}$$

and $v_{ij} = \sqrt{\beta\gamma}\ \varepsilon_{ij\cdot}$

The new variables are all independent having expectations zero and variances of the different sets are respectively $\Sigma_1^2 = \sigma^2 (1 - \rho_1)$, $\Sigma_2^2 = \sigma_2^2 (1 - \rho')$,

$$\Sigma_3^2 = \sigma_2^2 \{1 + (\beta - 1)\rho'\}$$

where $\sigma_2^2 = \sigma^2 \{1 + (\gamma - 1)\rho_1\}$, and $\rho' = \dfrac{\gamma \rho_2}{1 + (\gamma - 1)\rho_1}$

Also we have

$$\text{I} \quad \sum_{i=1}^{r} \sum_{j=1}^{\alpha} \sum_{k=1}^{\beta} \sum_{\lambda=1}^{\gamma-1} \eta_{ijk\lambda}^2 = r\alpha\beta \sum_{\lambda=1}^{\gamma-1} \eta^2_{\cdots\lambda} + r\beta \sum_j \sum_\lambda (\eta_{\cdot j\cdot\lambda} - \eta_{\cdots\lambda})^2$$

$$+ r \sum_k \sum_\lambda (\eta_{\cdot\cdot k\lambda} - \eta_{\cdots\lambda})^2 + r \sum_j \sum_k \sum_\lambda (\eta_{\cdot jk\lambda} - \eta_{\cdot j\cdot\lambda} - \eta_{\cdot\cdot k\lambda} + \eta_{\cdots\lambda})^2$$

$$+ \sum_i \sum_j \sum_k \sum_\lambda \eta_{ijk\lambda} - \eta_{\cdot jk\lambda})^2$$

$$\text{II} \qquad \sum_i \sum_j \sum_\nu \zeta^2_{ij\nu} = r\alpha \sum_{\nu=1}^{\beta-1} \zeta^2_{\cdot\cdot\nu} + r \sum_j \sum_\nu (\zeta_{\cdot j\nu} - \zeta_{\cdots\nu})^2$$

$$+ \sum_i \sum_j \sum_\nu (\zeta_{ij\nu} - \zeta_{\cdot j\nu})^2$$

III
$$\sum_i \sum_j v^2_{ij} = r\alpha v^2_{..} + \alpha \sum_i (v_{i\cdot} - v_{..})^2 + r \sum_j (v_{\cdot j} - v_{..})^2$$
$$+ \sum_i \sum_j (v_{ij} - v_{i\cdot} - v_{\cdot j} + v_{..})^2$$

Straightforward application of Cochran's rule gives the following analysis of Variance Table.

TABLE 4.4.1.

*Analysis of variance of Split Plot Design with r splits of subtreatments*

| Source | D F | S S |
|---|---|---|
| Blocks | $r-1$ | $\alpha\beta\gamma \sum_i (y_{i\cdots} - y_{\cdots\cdot})^2$ |
| Main treatments | $\alpha-1$ | $r\beta\gamma \sum_j (y_{\cdot j\cdot\cdot} - y_{\cdots\cdot})^2$ |
| Error $(a)$ | $(r-1)(\alpha-1)$ | $\beta\gamma \sum_i \sum_j (y_{ij\cdot\cdot} - y_{i\cdots} - y_{\cdot j\cdot\cdot} + y_{\cdots\cdot})^2$ |
| Subtreatments | $\beta-1$ | $r\alpha\gamma \sum_k (y_{\cdot\cdot k\cdot} - y_{\cdots\cdot})^2$ |
| Interaction : | | |
| Subtreatment × Main treatment | $(\alpha-1)(\beta-1)$ | $r\gamma \sum_j \sum_k (y_{\cdot jk\cdot} - y_{\cdot j\cdot\cdot} - y_{\cdot\cdot k\cdot} + y_{\cdots\cdot})^2$ |
| Error $(b)$ | $\alpha(r-1)(\beta-1)$ | $\gamma \sum_i \sum_j \sum_k (y_{ijk\cdot} - y_{ij\cdot\cdot} - y_{\cdot jk\cdot} + y_{\cdot j\cdot\cdot})^2$ |
| Sub-sub treatment | $\gamma-1$ | $r\alpha\beta \sum_l (y_{\cdots l} - y_{\cdots\cdot})^2$ |
| Main treatment × sub-sub treatment | $(\alpha-1)(\gamma-1)$ | $r\beta \sum_j \sum_l (y_{\cdot j\cdot l} - y_{\cdots l} - y_{\cdot j\cdot\cdot} + y_{\cdots\cdot})^2$ |
| Sub-treatment × sub-sub treatment | $(\beta-1)(\gamma-1)$ | $r\alpha \sum_k \sum_l (y_{\cdot\cdot kl} - y_{\cdot\cdot k\cdot} - y_{\cdots l} + y_{\cdots\cdot})^2$ |
| Main treatment × Sub-treatment × sub-sub-treatment | $(\alpha-1)(\beta-1)(\gamma-1)$ | $r\sum\sum\sum (y_{\cdot jkl} - y_{\cdot j\cdot l} - y_{\cdot\cdot kl} - y_{\cdot jk\cdot} + y_{\cdot j\cdot\cdot} + y_{\cdots l} + y_{\cdot\cdot k\cdot} - y_{\cdots\cdot})^2$ |
| Error $(c)$ | $\alpha\beta(r-1)(r-1)$ | $\sum\sum\sum\sum (y_{ijkl} - y_{\cdot jkl} - y_{ijk\cdot} - y_{\cdot jk\cdot})^2$ |

$$V(y_{..l}-y_{..l'}) = V\left[\sum_{\lambda=1}^{\gamma-1}(a_{l\lambda}-a_{l'\lambda})\eta_{..\lambda}\right]$$

$$=\frac{\Sigma_1^2}{r\alpha\beta}\left[2\left(1-\frac{1}{\gamma}\right)-2\left(-\frac{1}{\gamma}\right)\right]=2\Sigma_1^2/r\alpha\beta.$$

$$V(y_{..k\cdot}-y_{..k'\cdot}) = V\left[\sum_{\nu=1}^{\beta-1}(b_{k\nu}-b_{k'\nu})y_{..\nu}\right]=2\,\Sigma_2^2/r\alpha\gamma$$

Similarly $V(y_{\cdot j\cdot\cdot}-y_{\cdot j'\cdot\cdot})=2\Sigma_3^2/\alpha\beta\gamma,\ V(y_{\cdot\cdot kl}-y_{\cdot\cdot kl'})=2\,\Sigma_1^2/r\alpha$

$$V(y_{.j.l}-y_{.j'.l})=V\left[\sum_{\lambda=1}^{\gamma-1}a_{i\lambda}(\eta_{.j.\lambda}-\eta_{.j'.\lambda})\right]=\frac{2\Sigma_1^2}{r\beta}\left(1-\frac{1}{\gamma}\right)+\frac{2\Sigma_3^2}{r\beta\gamma}$$

$$V(y_{.j.l}-y_{.j.l'})=V\left[\sum_{\lambda=1}^{\gamma-1}(a_{l\lambda}-a_{l'\lambda})\eta_{.j.\lambda}\right]=2\Sigma_1^2/r\beta$$

$$V(y_{\cdot\cdot k'l}-y_{\cdot\cdot\cdot k'l})=V[y_{\cdot\cdot k\cdot}-y_{\cdot\cdot k'\cdot}+\Sigma a_{l\lambda}(a_{\cdot\cdot k\lambda}-\eta_{\cdot\cdot k'\lambda})]$$

$$=\frac{2\Sigma_2^2}{r\alpha\gamma}+\frac{2\Sigma_1^2(\gamma-1)}{r\alpha\gamma}$$

5. *Split-Plot with sub-units in Strips*. Here the subtreatments are arranged in strips across each replication. Let $y_{ijk}$ denote the yield of the plot having the $k$ th subtreatment, $j$ th main treatment in the $i$ th replication ($i=1,2,\ldots r$; $j=1,2,\ldots\ldots,\alpha$ $k=1,2,\ldots\ldots\beta$). Then the yields $y_{ijk}$, $y_{ij'k}$ and $y_{ijk}$ , $y_{ijk'}$ are correlated. Our set-up is

$$y_{ijk}=\mu+\alpha_i+\gamma_j+\tau_k+\delta_{jk}+\varepsilon_{ijk}$$
$$\text{with }\sum_i\alpha_i=\sum_j\gamma_j=\sum_k\tau_k=\sum_j\delta_{jk}=\sum_k\delta_{jk}=0$$

$$E(\varepsilon_{ijk})=0,\ V(\varepsilon_{ijk})=\sigma^2,\ \mathrm{Cov}(\varepsilon_{ijk},\ \varepsilon_{i'j'k'})=\rho_1\sigma^2$$

if $i=i'$, $j=j'$, $k\neq k'$ and $=\rho_2\sigma^2$ if $i=i'$, $k=k'$, $j\neq j'$ and $=0$ otherwise.

Let $(\varepsilon_{ijk})=\mathbf{A}\{\sqrt{\beta}\,\varepsilon_{ij\cdot}\ \ \eta_{ij1},\ \ldots\ldots,\ \eta_{ij\,\beta-1}\}$

$$\text{where }\mathbf{A}=\begin{bmatrix}1/\sqrt{\beta} & a_{11} & a_{12} & \ldots a_{1\beta-1}\\ 1/\sqrt{\beta} & a_{21} & a_{22} & \ldots a_{2\beta-1}\\ \ldots & \ldots & \ldots & \ldots\\ 1/\sqrt{\beta} & a_{\beta1} & a_{\beta2} & \ldots a_{\beta\,\beta-1}\end{bmatrix}$$

is an orthogonal matrix. Then the two sets of variates

I. $\eta_{ij\lambda}\ (i=1,2,..,r;j=1,2,..,\alpha;k=1,2,..,\beta-1)$

II. $u_{ij}=\sqrt{\beta}\ \varepsilon_{ij\cdot},\ (i=1,2,\ldots,r;j=1,2,\ldots,\alpha)$.

are independent in the probability sense. $\eta_{ij\lambda}$ 's have the same variance but $\eta_{ij\lambda}$ and $\eta_{ij'\lambda}$ are correlated. Similarly $u_{ij}$ and $u_{ij'}$ are correlated. Making use of a further transformation

$$(\eta_{ij\lambda}) = \begin{bmatrix} 1/\sqrt{\alpha} & b_{11} & b_{12} \cdots & b_{1\alpha-1} \\ 1/\sqrt{\alpha} & b_{21} & b_{22} \cdots & b_{2\alpha-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1/\sqrt{\alpha} & b_{\alpha_1} & b_{\alpha_2} \cdots & b_{\alpha\alpha-1} \end{bmatrix} \begin{bmatrix} \sqrt{\alpha}\, \eta_{i.\lambda} \\ \zeta_{il\lambda} \\ \cdots \\ \zeta_{i(\alpha-1)\lambda} \end{bmatrix} = B \begin{bmatrix} \sqrt{\alpha}\eta_{i.\lambda} \\ \zeta_{il\lambda} \\ \cdots \\ \zeta_{i(\alpha-1\lambda)} \end{bmatrix}$$

where $B$ is orthogonal and $(u_{ij}) = B \begin{bmatrix} \sqrt{\alpha}\, u_{i.} \\ v_i 1 \\ \cdots \\ v_{i(\alpha-1)} \end{bmatrix}$

We get the following sets of variates which are independent.

I. $v_{i\alpha} = \sqrt{\alpha}\, u_{i.} = \sqrt{\alpha\beta}\, \varepsilon_{i..}\ , \quad i = 1, 2, \ldots, r$

II. $v_{i\nu} = \sum_{j=1}^{\alpha} b_{j\nu}\ u_{ij} = \sqrt{\beta} \sum_{j=1}^{\alpha} b_{j\nu}\ \varepsilon_{ij.}, i = 1, 2, \ldots, r;\ \nu = 1, 2, \ldots, \alpha - 1.$

III. $\xi_{i\lambda} = \sqrt{\alpha}\, \eta_{i.\lambda} = \sum_{k=l}^{\beta} a_{k\lambda}\ \varepsilon_{i.k}\ ,\ i = 1, 2, \ldots, r;\ \lambda = 1, 2, \ldots \beta-1$

IV. $\zeta_{i\nu\lambda} = \sum_{j=1}^{\alpha} b_{j\nu}\ \eta_{ij\lambda} = \sum_{j=1}^{\alpha} \sum_{k=1}^{\beta} b_{j\nu}\ a_{k\lambda}\ \varepsilon_{ijk}$

$i = 1,2,\ldots, r;\ \nu = 1,2,\ldots \alpha-1;\ \lambda = 1,2,\ldots, \beta-1.$

Thus

$$\varepsilon_{ijk} = \frac{1}{\sqrt{\alpha\beta}}\, v_{i\alpha} + \frac{1}{\sqrt{\beta}} \sum_{\nu=1}^{\alpha-1} b_{j\nu}\, v_{i\nu} + \frac{1}{\sqrt{\alpha}} \sum_{\lambda=1}^{\beta-1} a_{k\lambda}\, \xi_{i\lambda} + \sum_{\lambda=1}^{\beta-1} \sum_{\nu=1}^{\alpha-1} a_{k\lambda}\ b_{j\nu}\ \zeta_{ij\lambda}$$

Now

$$\Sigma\Sigma v_{i\nu}{}^2 = r\Sigma v_{\cdot\nu}{}^2 + \sum_i\sum_\nu (v_{i\nu} - v_{\cdot\nu})^2$$

$$\Sigma\Sigma \xi_{i\lambda}{}^2 = r\sum_\lambda \xi_{\cdot\lambda}{}^2 + \sum_i\sum_\lambda (\xi_{\lambda i} - \xi_{\cdot\lambda})^2$$

$$\Sigma\Sigma\Sigma\zeta_{i\nu\lambda}{}^2 = r\Sigma\Sigma\zeta_{\cdot\nu\lambda}{}^2 + \Sigma\Sigma\Sigma(\zeta_{i\nu\lambda} - \zeta_{\cdot\nu\lambda})^2$$

Straightforward application of Cochran's rule gives the following analysis of Variance :

TABLE 4.5.1

*Analysis of Variance of Split-Plot Design with sub-unit treatment in strips*

| Source | D F | S S |
|---|---|---|
| Main Treatments | $\alpha - 1$ | $r\beta\Sigma(y_{\cdot j\cdot} - y_{\dots})^2$ |
| Error ($a$) | $(r-1)(\alpha-1)$ | $\beta\Sigma\Sigma(y_{ij\cdot} - y_{i\cdot\cdot} - y_{\cdot j\cdot} + y_{\dots})^2$ |
| Subtreatments | $\beta - 1$ | $r\alpha\Sigma(y_{\cdot\cdot k} - y_{\dots})^2$ |
| Error ($b$) | $(r-1)(\beta-1)$ | $\alpha\Sigma\Sigma(y_{i\cdot k} - y_{i\cdot\cdot} - y_{\cdot\cdot k} + y_{\dots})^2$ |
| Interaction : | | |
| Main Treatment $\times$ Sub-treatment | $(\alpha-1)(\beta-1)$ | $r\Sigma\Sigma(y_{\cdot jk} - y_{\cdot\cdot k} - y_{\cdot j\cdot} + y_{\dots})^2$ |
| Error ($c$) | $(r-1)(\alpha-1)(\beta-1)$ | $\Sigma\Sigma\Sigma(y_{ijk} - y_{i\cdot k} - y_{\cdot jk} - y_{ij\cdot} + y_{\cdot\cdot k} + y_{\cdot j\cdot} + y_{\dots} - y_{\dots})^2$ |

We can also establish very easily that

$$V(y_{\cdot j\cdot} - y_{\cdot j'\cdot}) = 2\sigma_1^2 / r\beta$$

$$V(y_{\cdot\cdot k} - y_{\cdot\cdot k'}) = 2\sigma_2^2 / r\alpha$$

$$V(y_{\cdot jk} - y_{\cdot j'k}) = 2[(\beta-1)\sigma_3^2 + \sigma_1^2] / r\beta$$

$$V(y_{\cdot jk} - y_{jk'}) = 2[(\beta-1)\sigma_3^2 + \sigma_2^2] / r\alpha$$

where $\sigma_1^2$, $\sigma_2^2$ and $\sigma_3^2$ can be calculated terms of $\sigma^2$, $\rho_1$ and $\rho_2$.

The analysis of sub-units in strips when main treatments are arranged in a Latin Square is exactly similar.

6. *Confounded Arrangement in Split-plot Designs.* Suppose that we have $p$ main treatments and $s^m$ subtreatments. Then we can have $ps$ whole plots and in a set of $s$ whole plots having one particular main treatment, we can split each main plot into $s^{m-1}$ sub-plots and assign to them the subtreatments in such a way that $(s-1)$ d.f. belonging to the highest order interaction of the

$m$ subtreatment factors are confounded with the whole plots. The subdivision of the degrees of freedom is as follows :

| | D.F. |
|---|---|
| Between Blocks | $b-1$ |
| Between Main Treatments | $p-1$ |
| Highest order Interaction of Subtreatment factors | $s-1$ |
| Interaction : | |
| Main Treatment $\times$ Highest order interaction of subtreatment factors | $(p-1)\ (s-1)$ |
| Error ($a$) | $(sp-1)\ (b-1)$ |
| Total | $bsp-1$ |
| Subtreatments | $s\ (s^{m-1}-1)$ |
| Main Treatment $\times$ Subtreatment | $s\ (s^{m-1}-1)\ (p-1)$ |
| Error ($b$) | $sp(b-1)\ (s^{m-1}-1)$ |
| Total | $bps^m-1$ |

Another method would be to take $s\ r$ blocks (of $p$ whole plots) consisting of $r$ sets of $s$ blocks. In any one set, the main plot is divided into $s^{m-1}$ subplots to accommodate $s^{m-1}$ of the $s^m$ subtreatments—different subsets of $s^{m-1}$ of the $s^m$ subtreatments being used for different members of a set. Here the subdivision of the $s^m$ subtreatments into $s$ sets of $s^{m-1}$ is such that interest contrasts belong to the highest order interaction of the subtreatment factors. The partitioning of the total degrees of freedom is as follows :

| Source | D.F. |
|---|---|
| Between blocks | $r\,s-1$ |
| Main Treatment | $p-1$ |
| Main Treatment $\times$ Highest order interaction of the subtreatment factors | $(p-1)\ (s-1)$ |
| Error ($a$) | $s\ (r-1)\ (p-1)$ |
| Total between whole plots | $rsp-1$ |
| Subtreatments | $s\ (s^{m-1}-1)$ |
| Main Treatment $\times$ Subtreatment | $s\ (s^{m-1}-1)\ (p-1)$ |
| Error ($b$) | $sp\ (r-1)\ (s^{m-1}-1)$ |
| Total | $r\ s^m\ p-1$ |

## Exercises to Chapter 4

1. Let $A_1 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $X_1 = \{1, x_1\}$ ; $A_i = \begin{bmatrix} A_{i-1} & A_{i-1} \\ -A_{i-1} & A_{i-1} \end{bmatrix}$

and $X_i = \{ X_{i-1}, X_i X'_{i-1} \}$, $i = 2, 3, \ldots, m$. Show that

$$A_i \; X_i = \begin{bmatrix} (x_i + 1) \; A_{i-1} \; X_{i-1} \\ (x_i - 1) \; A_{i-1} \; X_{i-1} \end{bmatrix}$$

Hence, establish Yates' process for obtaining main effects and interactions from total yields of treatment combinations in a $2^m$ design. Show that this can be used inversely to obtain the yields of treatment combinations from the total, main effects and interactions.

2. Show that Yates' method can be extended to designs in which some or all factors are at more than two levels.

Extend in particular Yates' method to $3^m$ design, taking

$$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix} ; \; X_1 = \{1, x_1, x_1^2\}$$

$$A_i = \begin{bmatrix} A_{i-1} & A_{i-1} & A_{i-1} \\ -A'_{i-1} & O & A_{i-1} \\ A_{i-1} & -2A_{i-1} & A_{i-1} \end{bmatrix} ; \; X_i = \{ X_{i-1}, x_i \; X_{i-1}, x_i^2 \; X_{i-1} \},$$

$i = 2, 3, \ldots, m$ and

noting

$$A_i \, X_i = \begin{bmatrix} (1 + x_i + x_i^2) & A_{i-1} & X_{i-1} \\ (-1 + x_i^2) & A_{i-1} & X_{i-1} \\ (1 - 2x_i + x_i^2) & A_{i-1} & X_{i-1} \end{bmatrix}$$

How will you arrange the treatment yields in order to make Yates' method operative ?

3. Deduce that in a $2^m$ factorial design replicated $r$ times in randomised blocks of $2^m$ plots, each main effect and interaction has variance $\sigma^2/r.2^{m-2}$ where $\sigma^2$ is the per plot intrablock variance.

4. Let

$$X_0(x) = 1,\ X_1(x) = x - \frac{s-1}{2},$$

$$X_t(x) = X_1(x)\,X_{t-1}(x) - (t-1)^2\{s_1^2-(t-1)^2\}X_{t-2}(x)/4(2t-1)(2t-3)$$

be orthogonal polynomials for the set of values $x = 0, 1, 2, \ldots\ldots\ldots s_1 - 1$. We can define for $t = 1, 2, 3, \ldots\ldots$

$$F_{1t} = \left\{ \sum_{x=0}^{s_1-1} f_1^x \; X_t(x) \right\} \phi_2\, \phi_3 \ldots\ldots \phi_m$$

as due to the linear, quadratic, cubic,.....effect of $F_1$. Show that the sum of squares due to the above effect in a randomised block experiment with $r$ replications is

$$\left[ Est\ F_{1t} \right]^2 \Big/ \frac{(t!)^4\, s_1\,(s_1^2-1^2)\ldots(s_1^2-t^2)}{r\, s_2\, s_3 \ldots s_m\,(2t)!\,(2t+1)!}$$

5. Three manures $N$, $P$, $K$ each at two levels are combined to form the eight manures $N_0P_0K_0 = a$, $N_1P_0K_0 = b$, $N_0P_1K_0 = c$, $N_0P_1K_1 = d$, $N_0P_0K_1 = e$, $N_1P_1K_0 = f$, $N_1P_0K_1 = g$, $N_1P_1K_1 = h$. Give the actual plan of an $8 \times 8$ Latin Square experiment to test the manures. Obtain estimates of the different main effects and interactions and calculate their variances.

6. Show that split-plot arrangements are specialised types of designs involving the confounding of main effects.

7. Show that when the main plot is split into two parts only, the analysis can be substantially simplified by calculations of sums and differences of adjoining sub-plot yields.

8. Show that the main effects, two, three and more factor interactions in a $2^m$ design are linear functions of the treatments (1), $a$, $b$, $ab$, $c$, $ac$, $bc$,, $abc$,—......with coefficients $+1$ or $-1$ according as the particular treatment contains an even or odd number of letters that appear in the factorial effect.

9. In the analysis of split plot design with $\alpha$ main treatments and $\beta$ sub-treatments in $r$ randomised blocks, arrange the observations $y_{ijk}$ as $Z_1, Z_2, \ldots Z_\lambda$, where $\lambda = (i-1)\alpha\beta + (j-1)\beta + k$. Write down the dispersion matrix $V$ and its inverse $V^{-1}$. Utilise the results to obtain the normal equations.

10. In Ex. 9, obtain best unbiased linear estimates of main treatment contrasts, subtreatment contrasts and utilise the normal equations to obtain their variances and covariances.

11. Obtain the expected values of sums of squares in Table 4.2.2 and Table 4.3.1.

12. Suppose that there are no main plot treatments and there are $p$ plots in an area. Divide each horizontally into $b$ strips to accommodate $b$ levels of treatment $B$. Independently of this divide the plots vartically into $c$ strips to accommodate $c$ levels of a treatment $C$. Write down the structure of analysis for this experiment, the partitioning of the total degrees of freedom and the variances of different treatment comparisons.

13. Indicate the change in the analysis when a set of main plot treatments, $A$, say at a levels, are imposed on to the Experiment in Ex. 12.

14. $\sigma_1$ and $\sigma_2$ are the standard errors associated with main-plot and sub-plot respectively, there being $\beta$ sub-plots per main plot. Establish that the variance between any two treatment means not belonging to the same main-plot involves the weighted average of $\sigma_1^2$ and $\sigma_2^2$, the weights being in the ratio $1 : (\beta - 1)$.

15. There are $\alpha$ main treatments where $\alpha$ is even, $2r$ blocks of $\alpha$ plots arranged in $r$ sets of 2. Each main plot is split into $2^{m-1}$ parts. The $2^m$ subtreatments are divided into two sets of $2^{m-1}$ such that the interest contrast belongs to the highest order interaction of the subtreatment factors. To $\alpha/2$ main plots of the first block of a set, we assign the first set of subtreatments, and the second set of subtreatments to the remaining main plots. In the second block of the set, we assign the subtreatment sets in the reverse order. Give the computational procedure of the arrangement and the partitioning of the total degrees of freedom.

16. Give the structure of the analysis of covariance for the split-plot design. Obtain the adjusted plot yields and the variances of different adjusted treatment comparisons.

17. A $3 \times 2 \times 3$ design is applied to the whole plots in blocks of 6 whole plots, with $AB$ and $ABC$ partially confounded and a factor $D$ at two levels is applied to the subplots. Give the computational procedure for the scheme that you would adopt.

18. Given a two-way whole-plot design with split-plot superimposed on one of the whole plot treatments. There are $s_1$ varieties and they are assigned to the columns randomly; in addition there are $s_2$ fertiliser levels which are assigned to the rows randomly. $s_3$ rates of seeding are to be tested and they are split-plot divisions of the varieties. There are $r$ replications. Obtain (*i*) the analysis of variance for the scheme (*ii*) the best unbiased linear estimates of defferent treatment comparisons together with their variances.

19. In a split-plot experiment with $r$ replications, $\alpha$ whole plots and $\beta$ sub-plots, take the yield of the plot in the first replication having first main treatment and first sub-treatment to be zero. Let $x_1$, the value of a concomitant variate, be $-1$ for this plot and zero elsewhere. Obtain the analysis of variance and covariance table and deduce the value of the error ($b$) regression coefficient.

20. Give the principle of construction and analysis of a design for a $4 \times 2^2$ factorial where the units are in a $4 \times 4$ Latin Square and the plots can be divided into two halves only, smaller units being impractical.

CHAPTER V

# APPLICATIONS OF GALOIS FIELDS AND FINITE GEOMETRY IN THE CONSTRUCTION OF DESIGNS

1. *Galois Fields.* A field $F$ is a set of more than one element for which there are defined operations of addition and multiplication which satisfy

Commutative Law : $a + b = b + a$, $ab = ba$

Associative Law : $a + (b + c) = (a + b) + c$, $a(bc) = (ab)c$

Distributive Law : $a(b + c) = ab + ac$.

Laws of Inverse Existence :

(*i*) For every pair $a, b$ there exists an $x$ such that $x + a = b$

(*ii*) For every pair $a, b$ satisfying the condition $a \neq 0$, there exists an $y$ such that $ya = b$.

A field containing a finite number of elements $s$ is called a Galois Field and is denoted by $GF(s)$. A Galois Field can always be constructed when $s = p^n$ where $p$ is a prime and $n$ is a positive integer. When $n = 1$, the residue classes modulo $p$ constitute $GF(p)$. In the general case the residue classes modulo $F(x)$ where $F(x)$ is a ploynomial in $x$ of degree $n$ which belongs to and is irreducible in $GF(p)$, constitute $GF(p^n)$. These functions $F(x)$ are known as minimum functions. Minimum functions likely to be of use in Design of Experiments are listed in Table 5.1.1 on page 80.

The elements of $GF(p^n)$ can also be expressed by $0, 1, \alpha, \alpha^2, \ldots \alpha^{p^n-2}$ where 0,1 and $\alpha$ denote respectively the null element, unit element and a primitive root of $x^{p^n-1} - 1 = 0$. Thus the nine elements of $GF(3^2)$ may be identified as $u_0 = 0$, $u_1 = 1$, $u_2 = \alpha$, $u_3 = \alpha^2 = \alpha + 1$, $u_4 = \alpha^3 = 2\alpha + 1$, $u_5 = \alpha^4 = 2$, $u_6 = \alpha^5 = 2\alpha$, $u_7 = \alpha^6 = 2\alpha + 2$ and $u_8 = \alpha^7 = \alpha + 2$. Now $u_5 + u_6 = 2 + 2\alpha + 2 = 2\alpha + 1 = u_4$ and $u_5 . u_6 = \alpha^4 . \alpha^5 = \alpha^9 = \alpha^2 = u_3$.

If $F(x) = a_0 + a_1 x + \ldots + a_n x^n$ where $a_0, a_1, a_2, \ldots, a_n$ are elements of $GF(p)$ is a minimum function for $GF(p^n)$, then it can be established that $x^t = (-1)^n a_0 \bmod F(x)$ where $t = \dfrac{p^n - 1}{p - 1}$. Thus knowing the series $0, 1, \alpha, \alpha^2, \ldots \alpha^{p^n} - 2$ up to $\alpha^{t-1}$ all others can be constructed by multiplication by $a_0$ and its powers.

TABLE 5.1.1

*List of Minimum Functions*

| $n$ \ $p$ | 2 | 3 | 5 | 7 | 11 | 13 |
|---|---|---|---|---|---|---|
| 2 | $x^2=x+1$ | $x^2=2x+1$ | $x^2=2x+2$ | $x^2=x-3$ | $x^2=4x-2$ | $x^2=x+1$ |
| 3 | $x^3=x+1$ | $x^3=x+2$ | $x^3=2x+3$ | $x^3=x-2$ | | |
| 4 | $x^4=x+1$ | $x^4=2x^3+2x^2+x+1$ | $x^4=x^3+x+2$ | $x^4=2x^3+2x+2$ | | |
| 5 | $x^5=x^2+1$ | $x^5=x+2$ | $x^5=x+2$ | $x^5=6x+3$ | | |
| 6 | $x^6=x+1$ | $x^6=x+1$ | $x^6=x^5-x^4+x^3-2x-2$ | | | |
| 7 | $x^7=x+1$ | | | | | |
| 8 | $x^8=x^4+x^3+x^2+1$ | | | | | |
| 9 | $x^9=x^8+x^4+x^3+x^2+x+1$ | | | | | |

2. *Finite Geometry.* The symbols $PG(k, p^n)$ and $EG(k, p^n)$ stand respectively for finite $k$-dimensional projective geometry and finite $k$-dimensional Euclidean Geometry.

In $PG(k, p^n)$, $(x_0, x_1, x_2, \ldots, x_k)$ where $x_0, x_1, x_2, \ldots, x_k$ are elements of $GF(p^n)$ at least one of which is different from zero will stand for a point. If $\rho \neq 0$ be an element of $GF(p^n)$, then $(x_0, x_1, x_2, \ldots, x_k)$ and $(\rho x_0, \rho x_1, \ldots \rho x_k)$ will stand for the same point. Thus there are $\dfrac{s^{k+1}-1}{s-1} = s^k + s^{k-1} + \ldots + 1$ points in our geometry, $s$ being written for $p^n$. All the points which satisfy the $(k-g)$ linearly independent equations.

$$a_{i0}\,x_0 + a_{i1}\,x_1 + \ldots + a_{ik}\,x_k = 0,\ i = 1,2,\ldots k-g\ldots \tag{5.2.1}$$

will be said to form a $g$-dimensional subspace or a $g$-flat in $PG(k,p^n)$. Evidently a $g$-flat is generated by $(g+1)$ points, the matrix of whose coordinates has rank $(g+1)$. The number of points in a $g$-flat is evidently $\dfrac{s^{g+1}-1}{s-1} = P_g$ say.

It is now easy to find out the number of $g$-flats in $PG(k,p^n)$, $g < k$. A $g$-flat is generated by $(g+1)$ linearly independent points. The first point can be chosen in $P_k$ ways. The second point in $P_k - P_0$ ways, the third point in $P_k - P_1$ ways and the $(g+1)$ th point in $P_k - P_{g-1}$ ways. Hence the total number of ways of choosing $(g+1)$ independent points $=$ $P_k(P_k - P_0)(P_k - P_1)\ldots(P_k - P_{g-1})$. Each $g$-flat can be generated by any one of $P_g(P_g - P_0)\ldots(P_g - P_{g-1})$, $(g+1)$ linearly independent points. Hence the number of $g$-flats is

$$\begin{aligned}\phi\ (k, s, g) &= \frac{P_k(P_k - P_0)\ldots(P_k - P_{g-1})}{P_g(P_g - P_0)\ldots(P_g - P_{g-1})}\\ &= \frac{(s^{k+1}-1)(s^k-1)\ldots(s^{k-g+1}-1)}{(s^{g+1}-1)(s^g-1)\ldots(s-1)}\end{aligned} \tag{5.2.2}$$

One can easily verify that

$$\phi\ (k, s, g) = \phi\ (k, s, k-g-1) \tag{5.2.3}$$

An ordered set of $k$ elements of $GF(p^n)$ $(x_1, x_2, \ldots x_k)$ will be called a point in $EG(k, p^n)$. Two points $(x_1, x_2, \ldots x_k)$ and $(x_1', x_2', \ldots, x_k')$ will be identical if and only if $x_i = x_i'$, $i = 1, 2, \ldots, k$. Thus there are $s^k$ points in our geometry, $s$ as usual standing for $p^n$. The set of points satisfying the consistent and independent $k-g$ equations.

$$a_{i0} + a_{i1}\,x_1 + a_{i2}\,x_2 + \ldots + a_{ik}\,x_k = 0,\ \ i = 1, 2, \ldots k-g \tag{5.2.4}$$

will be said to constitute a $g$-flat of $EG(k, p^n)$. Evidently the number of points in a $g$-flat is $s^g$.

From (5.2.1) and (5.2.4), it is obvious that $EG(k, p^n)$ may be derived from $PG(k, p^n)$ if we exclude from $PG(k, p^n)$ the points satisfying $x_0 = 0$ which constitute a $(k-1)$-flat in $PG(k, p^n)$. From (5.2.3) the number of $g$-flats in $EG(k, p^n)$ where $g < k$ is thus

$$\phi\ (k, s, g) - \phi\ (k-1, s, g) \tag{5.2.5}$$

3. *Orthogonal Latin Squares.* In section 2 of Chapter 3 we have already defined a Latin Square. If two Latin Squares with the same letters be such that when the two squares are superimposed each letter of one square coincides exactly once with each letter of the other square, then they are said to be orthogonal.

A set of Latin Squares, such that any two of them are mutually orthogonal will be said to constitute a set of orthogonal squares. If the Latin Squares be of side $s$, the number of squares in an orthogonal set can be at most $(s-1)$. To understand this, we first of all note that permutation of letters in the Latin Squares does not destroy the property of orthogonality. Thus we can re-write the squares of the set such that the first row in each is $AB$... Now between any two squares $B$ coincides with itself in the first row, hence in the first column $B$ cannot occupy any position twice or the first position. Thus there are at most $(s-1)$ positions available for $B$. When a set of $(s-1)$ orthogonalised Latin Squares of side $s$ exists, the set is said to be completely orthogonalised.

When $s = p^n$ where $p$ is a prime and $n$ is a positive integer, a set of completely orthogonalised Latin Squares can always be constructed.

Let $u_0 = 0$, $u_1 = 1$, $u_2 = \alpha$, $u_3 = \alpha^2$, ..., $u_{s-1} = \alpha^{s-2}$ be the elements of $GF$(s). Then in the $x$ th row and $y$ th column of the $i$ th Latin Square, we put the subscript of the element

$$u_i\, u_x + u_y$$

$$i = 1, 2, ..., s-1\ ;\ x = 0, 1, 2, ..., s-1, y = 0, 1, 2, ..., s-1.$$

A square so defined is evidently Latin. For if in the $x$ th row, the same number occurs twice, we must have $u_i\, u_x + u_y = u_i\, u_x + u_{y'}$, whence $u_y = u_{y'}$. Similarly if the same number occurs twice in the $y$ th column, we must have $u_i\ u_x + u_y = u_i\ u_{x'} + u_y$ whence $u_i\, u_x = u_i\, u_{x'}$ and since $u_i \neq 0$, $u_x = u_{x'}$. Again any two squares say the $i$ th and $i'$ th are orthogonal. For if in the $x$ th row and $y$ th column of $i$ th square we have $j$ and in the same position of the $i'$ th square we have $j'$, then

$$u_i\ u_x + u_y = u_j$$

$$u_{i'}\ u_x + u_y = u_{j'}$$

$$\therefore\ (u_i - u_{i'})\, u_x = u_j - u_{j'}$$

Since $u_i - u_{i'} \neq 0$, there is a unique solution $u_x$ and hence one and only one $u_y$. Thus the combination $(j, j')$ occurs once and only once.

The work of construction of the Latin Squares can be greatly simplified. The integer $j_i(x, y)$ to be put in the $x$ th row and $y$ th column of the $i$ th Latin Square is given by $u_{j_i(x,y)} = u_i u_x + u_y$. Hence if $x \neq 0, 1$ $j_i(x, y) = j_{i+1}(x-1, y)$, and $j_i(1, y) = j_{i+1}(s-1, y)$. Thus to get the complete set, we have to merely get the first Latin Square which is called the key Latin Square. Again it is easy to verify that for $1 \leqslant k \leqslant s-2$, $1 \leqslant k' \leqslant s-2$.

(i) $j_i(k+1, k'+1) = 0$ when $j_1(k, k') = 0$

(ii) $j_1(k+1, k'+1) = 1 + j_1(k, k')$ when $j_1(k, k') = 1, 2, \ldots, s-2$

(iii) $j_1(k+1, k'+1) = 1$ when $j_1(k, k') = s-1$

The 0-th row of the key Latin Square is $0, 1, 2, \ldots, s-1$ and we have merely to construct the 1st row. Knowing this row, we can write down the top right half of the square from the properties enumerated above. The other half can be written down from the property of symmetry.

Thus when $s = p^n$, $(s-1)$ orthogonal Latin Squares of side $s$ always exists. When $s$ is not a power of a prime, let $p_1^{e_1} p_2^{e_2} \ldots p_l^{e_l}$ be the canonical representation of $s$ as product of distinct primes. Let $r = \min\left(p_1^{e_1}-1, p_2^{e_2}-1, \ldots, p_l^{e_l}-1\right)$. Then we can show that a set of $r$ orthogonal Latin Squares of side $s$ exists. The proof is not difficult. Let $\alpha_i$ be a primitive element of $GF\left(p_i^{e_i}\right)$ and let $u_{i0} = 0$, $u_{i1} = 1$, $u_{i2} = \alpha_i$, $\ldots$, $u_{i\left(p_i^{e_i}-1\right)} = \alpha_i^{p_i^{e_i}-2}$ be the elements of $GF(p_i^{e_i})$. Consider the set of $s$ elements $u = (u_{1i_1}, u_{2i_2}, \ldots, u_{li_l})$. If $u^{(1)} = (u_{1i'_1}, u_{2i'_2}, \ldots u_{li'_l})$ and $u^{(2)} = u_{1i''_1}, \ldots, u_{1i''_l})$ be two elements of the set, we define $u^{(1)} + u^{(2)} = (u_{1i'_1} + u_{1i''_1}, \ldots, u_{li'_l} + u_{li''_l})$ and $u^{(1)}.\, u^{(2)} = u_{1i'_1}, u_{1i''_1}, \ldots, u_{li'_l}.\, u_{li''_l})$. The set constitutes a commutative ring and an element none of whose constituent coordinates is zero, possesses a unique inverse. Let us arrange the elements of the set so that the first $(r+1)$ elements of the set are $u_j = (u_{1j}, u_{2j}, \ldots, u_{lj})$, $0 \leqslant j \leqslant r$, the rest being numbered arbitrarily. In the $x$ th row and $y$ th column of the $j$ th square let us put the subscript of the element given by $u_j u_x + u_y$, $(x = 0, 1, 2, \ldots, s-1;\ y = 0, 1 \cdot 2 \ldots, s-1;\ j = 1, 2, \ldots, r)$. Note that for the considered values of $j$, $u_j$ possesses a unique inverse as also $u_j - u_{j'}$ if $j \neq j'$ and both lie between 1 and $r$. It is easy to establish now that the squares are Latin, and any two of the $r$ Latin Squares so constructed are orthogonal.

4. *Construction of BIBD.* The points of $PG(N, p^n)$ may be identified with varieties and the points lying on a $t$-flat, $1 \leqslant t \leqslant N$ may be said to constitute a block. Then $v = \dfrac{s^{N+1}-1}{s-1}$, $b = \phi(N, s, t) = \dfrac{(s^{N+1}-1)\ldots(s^{N-t+1}-1)}{(s^{t+1}-1)\ldots(s-1)}$,

$k$ = number of treatments per block $= \dfrac{s^{t+1}-1}{s-1}$, $r$ = number of replications = number of $t$-flats passing through a fixed point

$$= \frac{(P_N - P_0)(P_N - P_1)\dots(P_N - P_{t-1})}{(P_t - P_0)(P_t - P_1)\dots(P_t - P_{t-1})} = \frac{s^{t+1}-1}{s^{N+1}-1}.\ \phi(N, s, t) \text{ and}$$

$\lambda$ = the number of times a pair of treatments occur together in a block = the number of $t$ flats passing through a line

$$= \frac{(P_N - P_1)(P_N - P_2)\dots(P_N - P_{t-1})}{(P_t - P_1)(P_t - P_2)\dots(P_t - P_{t-1})} = \frac{(s^{t+1}-1)(s^t-1)}{(s^{N+1}-1)(s^N-1)}\ \phi(N, s, t)$$

*BIBD*'s can also be constructed from *EG* $(N, p^n)$. We identify as before the points of the geometry with varieties, and points in a $t$-flat where $1 \leqslant t \leqslant N$ with varieties in a block. The parameters of the design are $v = s^N$, $b = \phi(N, s, t) - \phi(N-1, s, t)$, $k$= number of points in a $t$-flat$=s^t$, $r$ = number of $t$-flats passing through a point

$$= \frac{(s^N-1)(s^{N-1}-1)\dots(s^{N-t+1}-1)}{(s^t-1)(s^{t-1}-1)\dots(s-1)}$$ and $\lambda$ = number of $t$-flats passing through a fixed pair of points $= \dfrac{(s^{N-1}-1)\dots(s^{N-t+1}-1)}{(s^{t-1}-1)\dots(s-1)}$.

From a given *BIBD*, it is always possible to derive a complementary design in the following way. In the $j$ th block of the complementary design, take all treatments which do not occur in the $j$ th block of the original. The parameters of the complementary design are $v' = v$, $b' = b$, $r' = b - r$, $k' = v - k$ and $\lambda' = b - 2r + \lambda$. By deleting a block and all treatments contained in this block we can derive from a symmetrical *BIBD* a new *BIBD* with parameters $v' = v - k$, $b' = b - 1$, $r' = r$, $k' = k - \lambda$ and $\lambda' = \lambda$. Again if in a symmetrical *BIBD*, we omit a block and retain in the remaining blocks treatments contained in the omitted block, we obtain a new *BIBD* with parameters $v' = k$, $b' = b - 1$, $r' = r - 1$, $k' = \lambda$ and $\lambda' = \lambda - 1$.

Let $u_0, u_1, \dots, u_{s-1}$ be the elements of *GF* (s) where $s = p^n$. Then if we can find $k$ elements $u_{i1}, u_{i2}, \dots, u_{ik}$ such that $k(k-1)$ differences $u_{i_l} - u_{i_m}$, $l \neq m$ are all the non-null elements of *GF*(s) each occurring exactly $\lambda = \dfrac{k(k-1)}{s-1}$ times, then we can construct a symmetrical *BIBD* with parameters $v=b=s$, $r = k$, $\lambda = \dfrac{k(k-1)}{s-1}$. The proof is trivial and the blocks can be obtained by adding to the initial block $(u_{i_1}, u_{i_2}, \dots, u_{i_k})$ the non-null elements of *GF*(s) in succession. Thus the differences among the elements 1, 3, 4, 5, 9 of *GF*(11) are 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10 each occurring twice. Hence (1, 3, 4, 5, 9), (2, 4, 5,

6, 10), (3, 5, 6, 7, 11), (4, 6, 7, 8, 1), (5, 7, 8, 9, 2), (6, 8, 9, 10, 3), (7, 9, 10, 11, 4), (8, 10, 11, 1, 5), (9, 11, 1, 2, 6), (10, 1, 2, 3, 7), (11, 2, 3, 4, 8) constitute the symmetrical *BIBD*: $v = b = 11$, $r = k = 5$, $\lambda = 2$. This is a particular case of a more general result : If $4\lambda + 3 = p^n$, then a symmetrical *BIBD* exists with parameters $v = b = 4\lambda + 3$, $r = k = 2\lambda + 1$ and $\lambda$. Let $\alpha$ be a primitive element of $GF\ (v)$. Then the elements of $GF\ (v)$ may be expressed as $0, 1, \alpha, \alpha^2, \ldots, \alpha^{v-2}$. Consider the set $(1, \alpha^2, \alpha^4, \ldots, \alpha^{4\lambda})$. The differences are $\pm(\alpha^{2i} - \alpha^0), \pm(\alpha^{2i+2} - \alpha^2), \ldots, \pm(\alpha^{4\lambda+2i} - \alpha^{4\lambda})$ $i = 1, 2, \ldots, \lambda$. Remembering that $\alpha^{2\lambda+1} = -1$ and writing $\alpha^{2i} - \alpha^0 = \alpha^{q_i}$, these differences can be rewritten as

$$\alpha^{q_i}, \alpha^{q_i+2}, \ldots, \alpha^{q_i+4\lambda}$$

$$\alpha^{q_i+2\lambda+1}, \alpha^{q_i+2\lambda+3}, \ldots, \alpha^{q_i+6\lambda+1}$$

These are the $4\lambda + 2$ non-null elements of $GF\ (4\lambda + 3)$. Hence every non-null element occurs exactly $\lambda$ times. Thus to construct the design $v = b = 19$, $r = k = 9$, $\lambda = 4$ we have merely to find the initial block. Now a primitive element of $GF(19)$ is 2. Hence the complete design can be constructed by adding 1, 2, 3, ..., 18 successively to the initial set (1, 4, 16, 7, 9, 17,11, 6, 5) and taking residues (mod. 19). By deleting this block and all treatments contained in it, we can construct the *BIBD* : $v = 10$, $b = 18$, $r = 9$, $k = 5$ and $\lambda = 4$.

5. *Construction of PBIBD*. $EG\ (N, s)$ and $PG\ (N, s)$ where $s = p^n$ may be employed for the construction of *PBIBD*. From $EG(N, s)$ we may omit the origin and all $m$-flats passing through it. If we take the retained $m$-flats as our blocks and the retained points as our varieties, we get a *PBIBD* with the following parameters.

$$v = s^N - 1$$

$$b = \phi\ (N, s, m) - \phi\ (N-1, s, m) - \phi\ (N-1, s, m-1)^*$$

$$r = \phi\ (N-1, s, m-1) - \phi\ (N-2, s, m-2)$$

$$k = s^m$$

$$n_1 = s^N - s \qquad \lambda_1 = \phi\ (N-2, s, m-2) - \phi\ (N-3, s, m-3)$$

$$n_2 = s - 2 \qquad \lambda_2 = 0$$

$$(p^1_{ij}) = \begin{pmatrix} s^N-2s+1 & s-2 \\ s-2 & 0 \end{pmatrix}, \ (p^2_{ij}) = \begin{pmatrix} s^N-s & 0 \\ 0 & s-3 \end{pmatrix}$$

$PG\ (N, s)$ can also be employed for the construction of *PBIBD*. We can omit the point (1 0 ... 0) and all $m$-flats passing through this point. Regarding

---

*For the sake of convenience we define $\phi\ (N, s, m) = 1$ if $m = -1$ and $\phi\ (N, s, m) = 0$ if $m \leqslant -2$.

the retained $m$-flats as blocks and retained points as varieties, we get a *PBIBD* with the following parameters.

$$v = \frac{s\,(s^N - 1)}{s-1}$$

$$b = \phi\,(N, s, m) - \phi\,(N-1, s, m-1)$$

$$k = (s^{m+1} - 1)\,/\,(s-1)$$

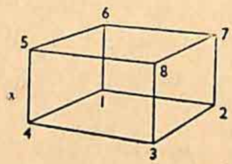
$$r = \phi\,(N-1, s, m-1) - \phi\,(N-2, s, m-2)$$

$$\lambda_1 = \phi\,(N-2, s, m-2) - \phi\,(N-3, s, m-3)$$

$$\lambda_2 = 0,\ n_1 = s^2\,(s^{N-1} - 1)\,/\,(s-1),\ n_2 = s-1$$

$$(p^1_{ij}) = \begin{pmatrix} n_1 - n_2 - 1 & n_2 \\ n_2 & 0 \end{pmatrix},\ (p^2_{ij}) = \begin{pmatrix} n_1 & 0 \\ 0 & n_2 - 1 \end{pmatrix}$$

Many *PBIBD* can be constructed by considering geometrical configurations. Thus we can get a *PBIBD* by taking the corners of a rectangular parallelopiped as varieties and faces as blocks :




Thus the blocks are (1, 2, 3, 4), (1, 4, 5, 6), (1, 6, 2, 7), (5, 6, 7, 8), (2, 3, 8, 7) and (3, 4, 5, 8). The parameters of the design are $v = 8$, $b = 6$, $r = 3$, $\lambda_1 = 0$, $n_1 = 1$, $\lambda_2 = 1$, $n_2 = 3$, $\lambda_3 = 2$, $n_3 = 3$ and

$$(p^1_{ij}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix} \quad (p^2_{ij}) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad (p^3_{ij}) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

If $\mathbf{A} = (a_{ij})$ is an $m \times n$ matrix and $\mathbf{B}$ *a* $p \times q$ matrix, then the Kronecker product of A and $B$ is defined as

$$\mathbf{A} \times \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & & a_{2n}\mathbf{B} \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & & a_{mn}\mathbf{B} \end{bmatrix}$$

which is an $mp \times nq$ matrix. If $\mathbf{N}_1$ and $\mathbf{N}_2$ be the incidence matrices of two *BIB* designs with parameters $v_1, b_1, r_1, k_1, \lambda_1$ and $v_2, b_2, r_2, k_2, \lambda_2$ then the design with incidence matrix $\mathbf{N} = \mathbf{N}_1 \times \mathbf{N}_2$ is a *PBIBD*. In fact when $r_1 \lambda_2 - r_2 \lambda_1 \neq 0$, the new design is a *PIBD* with three associate classes

having parameters $v' = v_1 v_2$, $b' = b_1 b_2$, $r' = r_1 r_2$, $k' = k_1 k_2$, $n'_1 = v_2 - 1$, $n'_2 = v_1 - 1$, $n'_3 = (v_1 - 1)(v_2 - 1)$, $\lambda'_1 = r_1 \lambda_2$, $\lambda'_2 = r_2 \lambda_1$, $\lambda'_3 = \lambda_1 \lambda_2$ and

$$(p^1_{ij}) = \begin{bmatrix} v_2 - 2 & 0 & 0 \\ 0 & 0 & (v_1 - 1) \\ 0 & (v_1 - 1) & (v_1 - 1)(v_2 - 2) \end{bmatrix},$$

$$(p^2_{ij}) = \begin{bmatrix} 0 & 0 & v_2 - 1 \\ 0 & v_1 - 2 & 0 \\ v_2 - 1 & 0 & (v_1 - 2)(v_2 - 1) \end{bmatrix},$$

$$(p^3_{ij}) = \begin{bmatrix} 0 & 1 & v_2 - 2 \\ 1 & 0 & v_1 - 2 \\ v_2 - 2 & v_1 - 2 & (v_1 - 2)(v_2 - 2) \end{bmatrix}.$$

Many interesting *PBIB* designs have been constructed in recent years by taking the Kronecker product of incidence matrices of known designs.

6. *Construction of Confounded Designs.* Let us consider the symmetrical factorial design in which there are $m$ factors $F_1, F_2, \ldots, F_m$ each at $s$ levels where $s = p^n$, a positive integral power of a prime. A treatment can be represented by $F_1^{x_1} F_2^{x_2} \ldots F_m^{x_m}$ where $x_1, x_2, \ldots x_m$ are respectively the levels of the different factors and may be identified with the point $(x_1, x_2, \ldots, x_m)$ of $EG\ (m, s)$. The effect of this treatment may be represented by $\tau(x_1, x_2, \ldots, x_m)$ or simply by $(x_1, x_2, \ldots x_m)$ also. Let the sum of the effects of the treatments satisfying $a_{11} x_1 + \ldots + a_{1m} x_m = \alpha_i$, $i = 0, 1, 2, \ldots, s-1$ where $\alpha_0, \alpha_1, \ldots \alpha_{s-1}$ are the elements of $GF(s)$ and $a_{11}, a_{12}, \ldots, a_{1m}$ are $m$ fixed elements of the same field, will be denoted by $\Sigma_{1i}$. The treatment contrast $l_{10} \Sigma_{10} + l_{11} \Sigma_{11} + \ldots + l_{1\,s-1} \Sigma_{1\,s-1}$ where $l_{10} + l_{11} + \ldots + l_{1\,s-1} = 0$ will be said to be a contrast carried by the pencil $P(a_{11}, a_{12}, \ldots, a_{1m})$. Evidently, there are $(s-1)$ such independent contrasts. If only $a_{1i_1}\ a_{1i_2} \ldots, a_{1i_k}$ are different from zero while all other $a$'s are zero, then in the contrasts belonging to the pencil there is symmetry with respect to the factors other than $i_1, i_2, \ldots, i_k$. Let $P(a_{21}, a_{22}, \ldots, a_{2m})$ be another pencil such that the rank of

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \end{bmatrix}$$

is 2. Then $a_{11} x_1 + a_{12} x_2 + \ldots + a_{1m} x_m = \alpha_i$ and $a_{21} x_1 + \ldots + a_{2m}\ x_m = \alpha_{i'}$, have $s^{m-2}$ points common. Hence if we consider the contrasts belonging to

these independent pencils $l_{10}\,\Sigma_{10} + l_{11}\,\Sigma_{11} + \ldots + l_{1s-1}\,\Sigma_{1s-1}$ and $l_{20}\,\Sigma_{20} + l_{21}\,\Sigma_{21} + \ldots + l_{2s-1}\,\Sigma_{2s-1}$ will have the sum of the product of their corresponding coefficients equal to

$$s^{m-2}\,(l_{10} + l_{11} + \ldots + l_{1s-1})\,(l_{20} + l_{21} + \ldots + l_{2s-1}) = 0.$$

Thus the contrasts of any pencil are orthogonal to the contrasts of any other independent pencil. It is, therefore, apparent that the $(s-1)$ degrees of freedom carried by the pencil $P\,(a_{11}, a_{12}, \ldots, a_{1m})$ in which only the elements $a_{1i_1}, \ldots, a_{1i_k}$ are different from zero, belong to the $(k-1)$ th order interaction of the factors $F_{i_1}, F_{i_2}, \ldots, F_{i_k}$. Evidently there are $(s-1)^{k-1}$ pencils in which these elements are different from zero and the rest are zero. These carry $(s-1)^k$ degrees of freedom belonging to the $(k-1)$th order interaction of $F_{i_1}, F_{i_2}, \ldots, F_{i_k}$. Let $a'_{i_j}$ s be elements of $GF\,(s)$ such that the rank of

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1m} \\ a_{21} & a_{22} & \ldots & a_{2m} \\ a_{k1} & a_{k2} & \ldots & a_{km} \end{bmatrix}$$

is $k$. Let us assign the treatments satisfying $a_{j1}\,x_1 + a_{j2}\,x_2 + \ldots + a_{jm}\,x_m = \alpha_{i_j}$, $j = 1, 2, \ldots, k$ to a block and let us designate this block as well as the sum of the treatment effects belonging to it by $\Sigma_{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}}$. Evidently this block contains $s^{m-k}$ treatments and there are $s^k$ such blocks. The contrasts

$$\sum_{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{ik}.} \; l_{\alpha_{i_1}, \alpha_{i_2} \cdots \alpha_{ik}} \; \Sigma_{\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{ik}}, \quad \Sigma\, l_{\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{ik}} = 0$$

are confounded with the blocks. They are carried by the $\dfrac{s^k - 1}{s-1}$ pencils

$$P\,(\lambda_1\,a_{11} + \lambda_2\,a_{21} + \ldots + \lambda_k\,a_{k1}, \ldots, \lambda_1\,a_{1m} + \lambda_2\,a_{2m} + \ldots + \lambda_k\,a_{km})$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are elements of GF$(s)$, not all zero, each having $(s-1)$ degrees of freedom.

For a representative contrast carried by the pencil

$$P(\lambda_1\,a_{11} + \lambda_2\,a_{21} + \ldots + \lambda_k\,a_{k1}, \ldots, \lambda_1\,a_{1m} + \ldots + \lambda_k\,a_{km})$$

can be written down as

$$l_0\,\Sigma'_0 + l_1\,\Sigma'_1 + \ldots + l_{s-1}\,\Sigma'_{s-1},\; l_0 + l_1 + \ldots + l_{s-1} = 0 \qquad (5.6.1)$$

where $\Sigma'_i$ is the sum of the effects of all treatments satisfied by $(\lambda_1\,a_{11} + \lambda_2\,a_{21} + \ldots + \lambda_k\,a_{k1})\,x_1 + \ldots + (\lambda_1\,a_{1m} + \ldots + \lambda_k\,a_{km})\,x_m = \alpha_{i-1}$ i. e. satisfied by $\lambda_1\,(a_{11}\,x_1 + a_{12}\,x_2 + \ldots + a_{1m}\,x_m) + \ldots + \lambda_k\,(a_{k1}\,x_1 + \ldots + a_{km}\,x_m) = \alpha_{i-1}$;

setting $a_{11} x_1 + a_{12} x_2 + \dots + a_{1m} x_m = \alpha_{i_1}, \dots a_{k1} x_1 + \dots + a_{km} x_m = \alpha_{i_k}$, we get $\lambda_1 \alpha_{i_1} + \lambda_2 \alpha_{i_2} + \dots + \lambda_k \alpha_{i_k} = \alpha_{i-1}$. Hence $\Sigma'_i = \Sigma (\Sigma_{\alpha_{i_1} \alpha_{i_2} \dots \alpha_{i_k}})$, the summation $\Sigma$ being over all $\alpha_{i_1}, \alpha_{i_2}, \dots \alpha_{i_k}$ such that $\lambda_1 \alpha_{i_1} + \dots + \lambda_k \alpha_{i_k} = \alpha_{i-1}$

Hence (5.6.1) is an interset contrast and is confounded with the blocks. The actual effects confounded can only be ascertained after knowing the number and positions of the non-null elements in $P(\lambda_1 a_{11} + \lambda_2 a_{21} + \dots + \lambda_k a_{k1}, \dots, \lambda_1 a_{1m} + \dots + \lambda_k a_{km})$.

7. *Hypercubes of Strength d.* Let $(x_1, x_2, \dots, x_m)$ be an ordered set of $m$ elements such that each $x_i$ can assume $s$ values. There are, therefore, $s^m$ such sets. A subset of $s^t$ of these sets will be said to constitute a hypercube of strength $d$ and will be denoted by $(m, s, t, d)$ if all combinations of any $d$ of the coordinates occur equally often, $s^{t-d}$ times. When $s = p^n$, a hypercube of strength $d$ can be constructed when we can find sets $(a_{i1}, a_{i2}, \dots, a_{it})$, $i = 1, 2, \dots, m$ where each $a_{ij}$ is an elements of $GF(s)$ such that any $d$ of them are independent. Let $y_1, y_2, \dots y_t$ each belong to $GF(s)$ and let $x_i = \sum_{j=1}^{t} a_{ij} y_j$. Then $(x_1, x_2, \dots, x_m)$ constitutes an $(m, s, t, d)$. For take the $i_1$ th, $i_2$ th, $\dots$, $i_d$ th coordinates. Then the equations

$$x_{i_j} = a_{i_j 1} y_1 + a_{i_j 2} y_2 + \dots + a_{i_j t} y_t$$
$$j = 1, 2, \dots, d$$

will be satisfied by $s^{t-d}$ distinct sets of values of $(y_1, y_2, \dots, y_t)$ and thus this particular combination of the $i_1$-th, $i_2$-th, $\dots$ $i_d$-th coordinates will occur $s^{t-d}$ times.

An $(m, s, t, d)$ can be utilised to construct an $(s^m, s^{m-t})$ design — *an* $s^m$ design in which a complete replication consists of $s^{m-t}$ blocks with $s^t$ treatments to each block—such that no main effect, first, second, $\dots$, $(d-1)$-th order interaction is confounded. For the array will contain the sets

$$(a_{1\lambda}, a_{2\lambda}, \dots, a_{m\lambda}), \qquad \lambda = 1, 2, \dots, t$$

and the contrasts carried by the pencil $P(b_1, b_2, \dots, b_m)$ will be confounded where

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1t} & a_{2t} & \cdots & a_{mt} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_m \end{bmatrix} = 0 .$$

Supposing that among $b_{i_1}, b_{i_2}, \dots, b_{i_d}$ there is one null element, while the rest are zero, it will mean

$$\begin{bmatrix} a_{i_1 1} & a_{i_2 1} & \dots & a_{i_d 1} \\ \dots & \dots & \dots & \dots \\ a_{i_1 t} & a_{i_2 t} & \dots & a_{i_d t} \end{bmatrix} \begin{bmatrix} b_{i_1} \\ b_{i_2} \\ b_{i_d} \end{bmatrix} = 0$$

i.e. the $i_1$-th, $i_2$-th, ..., $i_d$-th sets are dependent, contrary to hypothesis.
Example: Construct a $(3^4, 3^2)$ design such that the main effects and first order interactions are not confounded.

In $GF(3)$, the four sets (1, 0), (0, 1), (1, 1), and (2, 1) are such that any two of them are independent. Hence the contrasts carried by pencils $P(b_1, b_2, b_3, b_4)$ which are confounded are orthogonal to (1, 0, 1, 2) and (0, 1, 1, 1) and are therefore $P(2, 2, 1, 0)$, $P(1, 2, 0, 1)$ and their generalised interactions $P(2, 0, 1, 2)$ and $P(2, 1, 1, 0)$. The initial block is $(\lambda, \mu, \lambda + \mu, 2\lambda + \mu)$, $\lambda, \mu = 0, 1, 2$ and the other blocks can easily be constructed.

8. *Balancing.* In $(s^m, s^k)$ designs, the $\dfrac{s^k - 1}{s-1}$ sets of $(s - 1)$ degrees of freedom which are confounded in any one replication can be chosen at will. When, however, more than one replication is available, a better plan is not to confound the same degrees of freedom in each replication but to spread the loss of information as uniformly as possible among the interactions to be confounded. If each of the $(s - 1)^{k-1}$ sets of $(s - 1)$ degrees of freedom carried by the $(k - 1)$ th order interaction of the factors $F_{i_1}$ $F_{i_2}$ ..., $F_{i_k}$ is confounded in $r_1$ replications and remains unconfounded in $r_2$ replications, then we say that the interaction $F_{i_1} F_{i_2} \dots F_{i_k}$ has been balanced (there being uniform loss $r_1 / (r_1+r_2)$ on every degree of freedom belonging to this interaction).

Now there are ${}^m c_k$ $(k-1)$-th order interaction carrying in all ${}^m c_k\,(s-1)^{k-1}$ sets of $(s-1)$ degrees of freedom. If each of these sets is confounded in $r_1$ replications and remains inconfounded in $r_2$ replications, we say that a complete balance over the $(k-1)$-th order interaction has been achieved. Thus for a complete balance (*i*) every $(k-1)$ th order interaction has to be balanced and (*ii*) there should be equal loss of information on the different $(k - 1)$-th order interactions.

We can achieve complete balance over the highest order interaction in a $(s^m, s)$ design by taking $r = (s - 1)^{m-1}$ replications corresponding to the distinct pencils $P(a_1, a_2, \dots, a_m)$, $a_i \neq 0$, $i = 1, 2, \dots, m$ and assign to the key block the treatments satisfying—

$$a_1 x_1 + a_2 x_2 + \dots + a_m x_m = 0$$

Thus in $(3^3, 3)$ design, an arrangement achieving complete balance over second order interactions is given in the next page.

| Replication 1 $P(1, 1, 1)$ | | | Replication 2 $P(1, 2, 1)$ | | | Replication 3 $P(1, 2, 2)$ | | | Replication 4 $P(1, 1, 2)$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Block 1 | Block 2 | Block 3 | Block 4 | Block 5 | Block 6 | Block 7 | Block 8 | Block 9 | Block 10 | Block 11 | Block 12 |
| 0 0 0 | 1 0 0 | 2 0 0 | 0 0 0 | 1 0 0 | 2 0 0 | 0 0 0 | 1 0 0 | 2 0 0 | 0 0 0 | 1 0 0 | 2 0 0 |
| 1 1 1 | 2 1 1 | 0 1 1 | 0 1 1 | 1 1 1 | 2 1 1 | 2 2 0 | 0 2 0 | 1 2 0 | 1 1 2 | 2 1 2 | 0 1 2 |
| 1 0 2 | 2 0 2 | 0 0 2 | 0 2 2 | 1 2 2 | 2 2 2 | 1 1 0 | 2 1 0 | 0 1 0 | 2 2 1 | 0 2 1 | 1 2 1 |
| 1 2 0 | 2 2 0 | 0 2 0 | 1 0 2 | 2 0 2 | 0 0 2 | 1 0 1 | 2 0 1 | 0 0 1 | 1 0 1 | 2 0 1 | 0 0 1 |
| 2 2 2 | 0 2 2 | 1 2 2 | 1 1 0 | 2 1 0 | 0 1 0 | 2 1 1 | 0 1 1 | 1 1 1 | 1 2 0 | 2 2 0 | 0 2 0 |
| 2 1 0 | 0 1 0 | 1 1 0 | 1 2 1 | 2 2 1 | 0 2 1 | 0 2 1 | 1 2 1 | 2 2 1 | 0 1 1 | 1 1 1 | 2 1 1 |
| 2 0 1 | 0 0 1 | 1 0 1 | 2 0 1 | 0 0 1 | 1 0 1 | 0 1 2 | 1 1 2 | 2 1 2 | 0 2 2 | 1 2 2 | 2 2 2 |
| 0 1 2 | 1 1 2 | 2 1 2 | 2 1 2 | 0 1 2 | 1 1 2 | 2 0 2 | 0 0 2 | 1 0 2 | 2 1 0 | 0 1 0 | 1 1 0 |
| 0 2 1 | 1 2 1 | 2 2 1 | 2 2 2 | 0 2 2 | 1 2 2 | 1 2 0 | 2 2 2 | 0 0 2 | 2 0 2 | 0 0 2 | 1 0 2 |

To balance all interactions for designs of the class $(s^m, s^{m-1})$, we take $(x_1, x_2, \ldots, x_m)$ where each $x_i$ belongs to $GF(s)$ and is non-null. We construct the design with the help of the $(m-1)$ independent pencils $P(a_{i1}, \ldots, a_{im})$, $i = 1, 2, \ldots, m-1$ where $a_{i1}\, x_1 + a_{i2}\, x_2 + \ldots + a_{im}\, x_m = 0$. Evidently $(x_1, x_2, \ldots, x_m)$ and $\rho\,(x_1, x_2, \ldots, x_m)$ where $\rho$ is a non-null element of $GF(s)$ will give rise to the same set of pencils. Hence there will be $(s-1)^{m-1}$ replications and the constituents of the key-block will be $\alpha_i\,(x_1, x_2, \ldots, x_m)$, where $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{s-1}$ are the elements of $GF(s)$. No main effect is confounded. How many times will the $(s-1)$ degrees of freedom carried by the pencil $P\,(0\ldots, a_{i_1}, 0, \ldots, a_{i_2}, 0\ldots, a_{i_k}, 0, \ldots)$ be confounded? Eviently this number $r_k$ is equal to $(s-1)^{m-k}$ times the number of solutions of

$$a_{i_1} x_{i_1} + a_{i_2} x_{i_2} + \ldots + a_{i_k} x_{i_k} = 0$$

by distinct sets $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ none of which is zero. This latter number is

$$\frac{s^{k-1}-1}{s-1} - \binom{k}{1}\frac{s^{k-2}-1}{s-1} + \binom{k}{2}\frac{s^{k-3}-1}{s-1} \ldots + (-1)^{k-2}\binom{k}{k-2}\frac{s-1}{s-1}$$

$$= \frac{1}{s-1}\left\{\frac{1}{s}\left[s^k - \binom{k}{1}s^{k-1} + \ldots + (-1)^{k-2}\binom{k}{k-2}s^2\right] - \left[1 - \binom{k}{1} + \binom{k}{2} - \ldots + (-1)^{k-2}\binom{k}{k-2}\right]\right\}$$

$$= \frac{1}{(s-1)}\left\{\frac{(s-1)^k - (-1)^{k-1}ks - (-1)^k}{s} - [-(-1)^{k-1}k - (-1)^k]\right\}$$

$$= \frac{(s-1)^k + (-1)^k(s-1)}{s(s-1)} = \frac{(s-1)^{k-1} - (-1)^{k-1}}{s}.$$

Hence

$$r_k = (s-1)^{m-k} \quad \frac{(s-1)^{k-1} - (-1)^{k-1}}{s}.$$

Thus the loss of information on every $(s-1)$ degrees of freedom belonging to every $(k-1)$ -th order interaction is

$$\frac{(s-1)^{k-1} - (-1)^{k-1}}{s\,(s-1)^{k-1}}.$$

## Exercises to Chapter V

1. Construct Addition and Multiplication Tables for $GF(3^3)$ and $GF(4^2)$. Let $w$ be a primitive element of $GF(3^2)$. Writing the elements of the field as $0, \alpha_1, \ldots, \alpha_8$ where $\alpha_i = w^{i-1}$, $i = 1, 2, 3, \ldots, 8$, $(i)$ obtain the value of $-\alpha_3^2 + (\alpha_2\alpha_4 + \alpha_3\alpha_5)/\alpha_4$ $(ii)$ solve the equations $\alpha_3 x + \alpha_2 y = \alpha_1$ and $\alpha_5 x + \alpha_7 y = \alpha_3$.

2. A non-null element of a field is called a quadratic residue if it is the square of some element of the field, otherwise it is called a quadratic non-residue. When $s = p^n$, $p > 2$ show that the number of residues is equal to the number of non-residues in $GF(s)$. Defining $\chi(0), =0$, $\chi(a) = +1$ or $-1$ according as $a$ is a quadratic residue or not, show that $\sum\limits_{v} \chi(v^2 - 1)$ $= -1$, where $v$ runs over all elements of $GF(s)$.

3. Let $2u + 1 = p^n$ where $p$ is a prime, and $x$ be a primitive element of $GF(p^n)$. Show that among the $u(u-1)$ differences of the elements $x^0$, $x^2$, $\ldots$, $x^{2u-2}$ (*i*) every non-null element occurs exactly $(u-1)/2$ times if $u$ be odd (*ii*) every non-square element occurs $u/2$ times and every non-null square element occurs $(u-2)/2$ times when $u$ is even.

4. Construct two orthogonal Latin Squares of side 12.

5. Number the rows and columns as 0, 1, 2, ..., $s-1$. Obtain row 1 of the Key Latin Square from the Addition Table of $GF(s)$. To fill up the remaining rows proceed as follows: Take any element in row 1 and proceed by 1 single step in the direction of the leading diagonal. If the initial number in row 1 is 0, fill each successive cell by zero; if it is other than 0, put in each successive cell one greater than the number in the preceding cell, remembering that when $(s-1)$ is reached, the next cell is to be filled up by 1. The remaining portion of the key Latin Square can be filled up easily as the square is symmetrical about the leading diagonal. The other squares are obtained by cyclic interchange of columns leading 1, 2, ..., $(s-1)$.

Do the actual construction for $s = 4$, 8, 9, 16, 25 and 27.

6. How may $c$ married couples play a $(c-1)$-round bridge tournament, if each man plays one round with each lady except his wife and everybody plays with everybody else except his or her spouse. Show that a necessary condition for a solution to exist is that there are three $c \times c$ Latin Squares, orthogonal in pairs. Show further that when $c = 2^n$, $n > 1$, a solution always exists.

7. Using orthogonal Latin Squares of side $s$ where $s$ is a power of a prime, construct the following incomplete block designs:

(*i*) $v = s^2$, $b = s^2 + s$, $r = s + 1$, $k = s$ and $\lambda = 1$

(*ii*) $v = b = s^2 + s + 1$, $r = k = s + 1$, $\lambda = 1$.

8. Associate varieties with residue classes mod v. Let the elements of the the sets $(a_{1i}, a_{2i}, \ldots, a_{ki})$, $i = 1, 2, \ldots, t$ be residue classes mod $v$ such that (*i*) the members of each set are distinct (*ii*) among the $tk(k-1)$ intraset differences, each non-zero element occurs $\lambda$ times. Then $(a_{1i} + \theta, a_{2i} + \theta, \ldots, a_{ki} + \theta)$ where $i = 1, 2, \ldots, t$ and $\theta$ runs over all residue classes mod $v$, constitute a *BIBD* with parameters $b = vt$, $v$, $r = kt$, $k$ and $\lambda$.

9. Use Ex. 8 to construct (*i*) the *BIBD* : $v = 9$, $b = 18$, $r = 8$, $k = 4$ and $\lambda = 3$ from the initial sets (0, 1, 2, 4), (0, 3, 4, 7) ; (*ii*) the *BIBD* with $v$ varieties where $v$ is a prime power and of the form $6t + 1$, $b = vt$, $r = 3t$, $k = 3$ and $\lambda = 1$ from the initial set $(x^{i-1}, x^{2t+i-1}, x^{4t+i-1})$, $i = 1, 2, \ldots, t$ and $x$ is a primitive element of $GF(v)$. Construct in particular the *BIBD* : $v = 13$, $b = 26$, $r = 6$, $k = 3$ and $\lambda = 1$.

10. Use Finite Geometry to construct the following *BIBD* :

(*i*) $v = b = 15$, $r = k = 7$, $\lambda = 3$

(*ii*) $v = 8$, $b = 14$, $r = 7$, $k = 4$, $\lambda = 3$

(*iii*) $v = 15$, $b = 35$, $r = 7$, $k = 3$, $\lambda = 1$.

11. Show that a necessary condition for a symmetrical *BIBD* with an odd number of varieties to exist is that $x^2 \equiv \lambda \bmod p$ is solvable when $v = 4t + 1$ and $x^2 + \lambda \equiv 0 \bmod p$ is solvable when $v = 4t + 3$ where $p$ is a prime dividing the square free part of $(r - \lambda)$. Hence show that it is not possible to construct the following *BIBD* :

(*i*) $v = b = 43$, $r = k = 15$, $\lambda = 5$

(*ii*) $v = b = 29$, $r = k = 8$, $\lambda = 2$

(*iii*) $v = b = 67$, $r = k = 12$, $\lambda = 2$

(*iv*) $v = b = 77$, $r = k = 20$, $\lambda = 5$.

12. A school mistress is in the habit of taking her fifteen girls for a daily walk and they are arranged in five rows of three each so that each girl might have two companions. Give an arrangement so that for seven consecutive days no girl will walk with any one of her school fellows in any triplet more than once.

13. Let $(a_{1i}, a_{2i}, \ldots, a_{ki})$: $i=1, 2, \ldots, t$ be $t$ sets whose elements are residue classes mod $v$. Let (*i*) among the $kt\,(k-1)$ intraset differences, the non-nuil elements $\alpha_1^{(i)}, \alpha_2^{(i)}, \ldots, \alpha_{n_i}^{(i)}$ each occur $\lambda_i$ times, $i=1, 2, \ldots, m$. (*ii*) among the $n_i\,(n_i-1)$ differences $\alpha_u^{(i)}-\alpha_w^{(i)}$ $(u, w=1, 2, \ldots, n_i, u\neq w)$, each of $\alpha_1^{(l)}, \alpha_2^{(l)}, \ldots,$ $\alpha_{n_i}^{(l)}$ occur $p_{ii}^l$ times (*iii*) among the $n_i\, n_j$ difference, $\alpha_u^i-\alpha_x^j$, $(u=1, 2, \ldots, n_i,$ $x=1, 2, \ldots, n_j)$ the numbers $\alpha_1^{(l)}, \alpha_2^{(l)}, \ldots, \alpha_{n_i}^{(l)}$ each occur $p_{ij}^l$ times, $l=1, 2, \ldots, m$. Show that $(a_{1i}+\theta, a_{2i}+\theta, \ldots, a_{ki}+\theta)$, $i=1, 2, \ldots, t$ and $\theta$ running over all elements mod $v$ will constitute a *PBIB* with parameters $vt$, $v$, $tk$, $k$, $\lambda_1$, $\lambda_2 \ldots, \lambda_m$ and $(p_{ij}^l)$, $i, j, l = 1, 2, \ldots, m$.

14. Use Ex. 13 to construct the *PBIB* : $v = 17$, $b = 38$, $r = 8$, $k = 4$, $n_1 = n_2 = 8$, $\lambda_1 = 1$, $\lambda_2 = 2$, $(p_{ij}^1) = \binom{3\ 4}{4\ 4}$ and $(p_{ij}^2) = \binom{4\ 4}{4\ 3}$ from the initial blocks (0, 2, 9, 11), (0, 1, 4, 5).

15. $p^2 \times q$ varieties are arranged as a three dimensional lattice of points, $p$ along $x$ and $y$-axes and $q$ along $z$-axis. If blocks are formed consisting of all treatments represented by points lying in planes parallel to the $x\ z$ and $y\ z$ coordinate planes, show that the resulting design is a *PBIBD*. Obtain the paramaters of the design and show that its efficiency factor is

$$(p^2q-1) / [(p^2q-1)+2(p-1)].$$

16. Use Finite Geometry to construct the following *PBIBD* :

(*i*) $v = 14,\ b = 28,\ r = 6,\ k = 3$
$\lambda_1 = 1,\ n_1 = 12,\ \lambda_2 = 0,\ n_2 = 1,\ p^1{}_{11} = 10$

(*ii*) $v = b = 12,\ r = k = 6,\ \lambda_1 = 3,\ n_1 = 8,\ \lambda_2 = 2,\ n_2 = 3,\ p^1{}_{11} = 4.$

(*iii*) $v = b = 36,\ r = k = 5,\ \lambda_1 = 1,\ n_1 = 20,\ \lambda_2 = 0,\ n_2 = 15,\ p^1{}_{11} = 10.$

17. Associate two varieties to each element $0, 1, x, x^2, \ldots, x^{2u-1}$ of $GF(s)$ where $s = 2u + 1 = p^n$, $p \geqslant 3$ and $x$ is a primitive element, by means of lower suffixes 1 and 2. Show that by adding the elements of $GF(s)$ in succession to the constituents of the initial blocks

$$(0_1, 1_1, x_1^2, x_1^4, \ldots, x_1^{2u-2}, 1_2, x_2^2, \ldots, x_2^{2u-2})$$
$$(1_1, x_1^2, x_1^4, \ldots, x_1^{2u-2}, 0_2, 1_2, x_2^2, \ldots, x_2^{2u-2})$$

we obtain a *PBIBD* with parameters $v = b = 2s$, $r = k = s$,

$$\lambda_1 = (s-1)/2,\ \lambda_2 = s-1,\ n_1 = 2s-2,\ n_2 = 1,\ p_{11}^1 = 2s-4.$$

18. In a $2^m$ factorial experiment in $2^k$ blocks of $2^{m-k}$ plots each, if the main effects are denoted by $A$, $B$, $C$, ..., the two factors interactions by $AB$, $BC$, ..., and so on, show that if any two interactions are chosen to confound, the third is automatically determined by throwing together the letters of the two already chosen and suppressing any letters they have in common. Show further that the constituents of the key block are treatments having an even number of letters in common with the confounded effects and the constituents of the other blocks can be obtained by multiplying the constituents of the initial block by treatments not present in it and suppressing any letters they have in common.

19. Construct (*i*) a $2^5$ design in blocks of 8 plots confounding *ABC*, *ADE* and *BCDE* (*ii*) a $2^6$ design in blocks of 8 plots confounding *ABC*, *CDE*, *ADF*, *BEF*, *ABDE*, *BCDF* and *ACEF* (*iii*) a $2^6$ design in blocks of 16 plots confounding *ABCD*, *ABEF*, *CDEF*.

20. Establish that in $s^m$ design where $s$ is a power of a prime and the block size is $s^k$, the maximum number of factors which can be accommodated so that no main effect or two factors interactions are confounded is $(s^k-1)/(s-1)$.

21. Construct a $2^{15}$ design in blocks of 16 plots so that main effects or two factors interactions are not confounded.

22. Show that using blocks of $s^r$ plots ($s = p^n$, where $p$ is a prime), it is possible to test all combinations of so many as $N_k$ (but not more) factors in such a way that any interaction confounded does not involve less than $k$ factors, if $N_k$ denotes the maximum number of distinct points which can be chosen in $PG(r-1, p^n)$ such that no $(k-1)$ of the chosen points is conjoint.

Show in particular that for an experiment in which each factor is at two levels using blocks of $2^r$ plots it is possible to accommodate a maximum number of $2^{r-1}$ factors, if confounded interactions do not involve less than four factors.

23. Construct (*i*) a confounded $3^3$ design in 9 blocks of 3 plots in which four of the degrees of freedom confounded are carried by the pencils $P(1, 1, 0)$ and $P(1, 1, 1)$. (*ii*) a confounded $4^3$ design in 16 blocks of 4 plots each in which six of the degrees of freedom are carried by the pencils $P(1, 1, 1, 1)$ and $P(1, 1, 1, \alpha)$ where $\alpha$ is a primitive element of $GF(4)$.

24. For a $3^3$ design in nine blocks of three plots each, the constitution of the blocks are as follows: (000, 011, 022), (101, 112, 120), (202, 210, 221), (001, 012, 020), (102, 110, 121), (200, 211, 222), (002, 010, 021), (100, 111, 122), (201, 212, 220). Which 8 degrees of freedom are confounded?

25. Show that the total number of systems of confounding for a $s^m$ experiment in blocks of $s^k$, $s$ being a power of prime, is

$$\frac{(s^m-1)(s^{m-1}-1)\ldots(s^{m-k+1}-1)}{(s^{m-k}-1)(s^{m-k-1}-1)\ldots(s-1)}.$$

26. Show that in the case of $s^4$ design in $s^3$ blocks of $s$ plots each, to achieve a complete balance over the first and second order interactions $(s-1)^2$ replications are sufficient.

27. Construct 8 replications of a $3^4$ design in 3 plot blocks achieving complete balance over the first, second and third order interactions with a loss of half of information on the first order interactions, a loss of quarter of information on the second order interactions and a loss of 3/8 of information on the third order interactions.

Show that when 8 replications are not available, we can achieve a complete balance over the first and second order interactions in 4 replications with a loss of half of information on the first order interactions and 1/4 of information on second order interactions.

28. Construct 9 replications of the $4^3$ design in 4 plot blocks achieving a complete balance over the first and second order interactions. Calculate the loss of information on these interactions.

29. For the $s^3$ design in $s^2$ blocks of $s$ plots each, show that in order to achieve balance over first order interactions only, the number of replications necessary is $(s - 1)$ when $s$ is the power of an odd prime and $2(s - 1)$ when $s$ is a power of 2.

30. Show that if we take $(s - 1)^2$ replications of a $s^4$ design in $s^2$ blocks of $s^2$ plots each in which the key block of a particular replication contains the treatments

$$x_1 + \alpha x_3 + \beta x_4 = 0$$
$$x_2 + \beta^{-1} x_3 + d\alpha^{-1} x_4 = 0$$

where $c$, $d$ are fixed non-null elements of $GF(s)$, $s = p^n$ and $\alpha$ and $\beta$ are all possible non-null values, a complete balance over the second order interaction is achieved.

31. Construct 3 replications of a $4^3$ design in 16 blocks of 4 plots each achieving complete balance over the first order interaction with a loss of 1/3 of information.

CHAPTER VI

# SOME SELECTED TOPICS IN DESIGNS OF EXPERIMENTS

1. *Missing Plot Technique.* Often in experimental work, one or more experimental units may be missing through accident. The experimental design with which we started thus gets modified. The usual Least Square method can be applied to meet the new situation. The missing plot technique, however, seeks to provide numbers in the missing plots so that the ordinary procedures of the original design can be used.

Suppose that $y_1, y_2, \ldots, y_n$ are the observed yields in the existent plots and $x_1, x_2, \ldots, x_k$ are the yields that might have come from the missing plots. Usually, the expectations of these yields are linear functions of some parameters $\theta_1, \theta_2, \ldots, \theta_m$. We give below the ordinary least square analysis and the analysis with the substitution $x_1, x_2, \ldots, x_k$ in the missing plots.

(*a*) Analysis with existent observations

$$E(\mathrm{y}) = E\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11}\,a_{12}\cdots\cdots a_{1m} \\ a_{21}\,a_{22}\cdots\cdots a_{2m} \\ \cdots\cdots\cdots\cdots\cdots \\ a_{n1}\,a_{n2}\cdots\cdots a_{nm} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta \\ \cdots \\ \theta_m \end{bmatrix} = \mathrm{A}\boldsymbol{\theta}\ldots \qquad (6.1.1\text{a})$$

Sum of Squares to be minimised =

$$(\mathrm{y} - \mathrm{A}\,\boldsymbol{\theta})'\,(y - \mathrm{A}\,\boldsymbol{\theta}) = \mathrm{y}'\,\mathrm{y} - 2\boldsymbol{\theta}'\,\mathrm{A}'\,\mathrm{y} + \boldsymbol{\theta}'\,\mathrm{A}'\,\mathrm{A}\,\boldsymbol{\theta} \qquad (6.1.2\text{a})$$

Differentiating (6.1.2a) with respect to $\boldsymbol{\theta}$ and equating to zero, we get the normal equations

$$\mathrm{A}'\,\mathrm{y} = \mathrm{A}'\,\mathrm{A}\,\hat{\boldsymbol{\theta}} \qquad (6.1.3\text{a})$$

The residual sum of squares is equal to

$$\mathrm{y}'\,\mathrm{y} - \hat{\boldsymbol{\theta}}'\,\mathrm{A}'\,\mathrm{y} \qquad (6.1.4\text{a})$$

Suppose that we want to test the hypothesis $\theta_{l+1} = \theta_{l+2} = \ldots = \theta_m = 0$. Then we have to calculate the residual sum of squares with $\theta_{l+1} = \theta_{l+2} = \ldots = \theta_m = 0$. The difference between this conditional sum of squares and (6.1.4a) will be the sum of squares due to hypothesis to be tested against (6.1.4a). To obtain this conditional residual sum of squares, we proceed as follows :—

$$E(\mathrm{y}) = \begin{bmatrix} a_{11}\,a_{12}\cdots a_{11} \\ \cdots\cdots\cdots\cdots \\ a_{n1}\,a_{n2}\cdots a_{n1} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \cdots \\ \theta_l \end{bmatrix} = \mathrm{A}_1\,\boldsymbol{\phi} \qquad (6.1.5\text{a})$$

Sum of squares to be minimised is equal to

$$(\mathbf{y} - \mathbf{A}_1 \boldsymbol{\phi})' (\mathbf{y} - \mathbf{A}_1 \boldsymbol{\phi}) \tag{6.1.6a}$$

The normal equations are

$$\mathbf{A}'_1 \mathbf{y} = \mathbf{A}'_1 \mathbf{A}_1 \hat{\boldsymbol{\phi}} \tag{6.1.7a}$$

$\therefore$ Conditional residual sum of squares is

$$\mathbf{y}' \mathbf{y} - \hat{\boldsymbol{\phi}}' \mathbf{A}_1' \mathbf{y} \tag{6.1.8a}$$

(*b*) Analysing with substitutions of $x_1, x_2, \ldots, x_k$ in the missing plots.

$$E(\mathbf{y}) = \mathbf{A}\boldsymbol{\theta}, E(\mathbf{x}) = E \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} = \begin{bmatrix} b_{11}\, b_{12} \ldots b_{lm} \\ \ldots\ldots\ldots\ldots \\ b_{k1}\, b_{k2} \ldots b_{km} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \ldots \\ \theta_m \end{bmatrix} = \mathbf{B}\boldsymbol{\theta} \tag{6.1.1b}$$

Sum of squares to be minimised is equal to

$$(\mathbf{y} - \mathbf{A}\boldsymbol{\theta})' (\mathbf{y} - \mathbf{A}\boldsymbol{\theta}) + (\mathbf{x} - \mathbf{B}\boldsymbol{\theta})' (\mathbf{x} - \mathbf{B}\boldsymbol{\theta}) \tag{6.1.2b}$$

Normal equations are

$$\mathbf{A}' \mathbf{y} + \mathbf{B}' \mathbf{x} = (\mathbf{A}' \mathbf{A} + \mathbf{B}' \mathbf{B}) \hat{\hat{\boldsymbol{\theta}}} \tag{6.1.3b}$$

i.e.

$$\mathbf{y}' \mathbf{A} + \mathbf{x}' \mathbf{B} = \hat{\hat{\boldsymbol{\theta}}}' (\mathbf{A}' \mathbf{A} + \mathbf{B}' \mathbf{B}) \tag{6.1.4b}$$

$$\mathbf{B} = \frac{d\hat{\hat{\boldsymbol{\theta}}}'}{d\mathbf{x}} (\mathbf{A}' \mathbf{A} + \mathbf{B}' \mathbf{B}) \tag{6.1.5b}$$

Residual Sum of squares is equal to $S_e^2$ where

$$S_e^2 = \mathbf{y}' \mathbf{y} + \mathbf{x}'\mathbf{x} - \hat{\hat{\boldsymbol{\theta}}}' (\mathbf{A}' \mathbf{y} + \mathbf{B}' \mathbf{x}) \tag{6.1.6b}$$

Differentiating $S_e^2$ with respect to $\mathbf{x}$ and equating the derivative to zero, we get

$$2\mathbf{x} - \frac{d\hat{\hat{\boldsymbol{\theta}}}'}{d\mathbf{x}} (\mathbf{A}' \mathbf{y} + \mathbf{B}' \mathbf{x}) - \mathbf{B}\hat{\hat{\boldsymbol{\theta}}} = 0 \tag{6.1.7b}$$

Remembering (6.1.5b), $\mathbf{x} = \mathbf{B}\hat{\hat{\boldsymbol{\theta}}}$ (6.1.8b)

Substituting (6.1.8b) in (6.1.3b), we get the same normal equations as in (6.1.3a) and substituting (6.1.8b) in (6.1.6b), we get the same residual sum of squares as in (6.1.4a). Hence we have Fisher's rule :

If variates $x_1, x_2, \ldots, x_k$ be missing, we can proceed as if they are present provided that we put for them values which would minimise the residual sum of squares. The residual sum of squares so obtained is the actual residual sum of squares.

Let us find out the conditional residual sum of squares. Here the sum of squares to be minimised is

$$(\mathbf{y} - \mathbf{A}_1 \boldsymbol{\phi})' (\mathbf{y} - \mathbf{A}_1 \boldsymbol{\phi}) + (\mathbf{x} - \mathbf{B}_1 \boldsymbol{\phi})' (\mathbf{x} - \mathbf{B}_1 \boldsymbol{\phi}) \tag{6.1.9b}$$

The normal equations in the case, are

$$(\mathbf{A}'_1\,\mathbf{y} + \mathbf{B}'_1\,\mathbf{x}) = (\mathbf{A}'_1\,\mathbf{A}_1 + \mathbf{B}'_1\,\mathbf{B}_1)\,\hat{\hat{\boldsymbol{\phi}}} \tag{6.1.10b}$$

whence

$$\mathbf{B}_1 = \frac{d\,\hat{\hat{\boldsymbol{\phi}}}}{d\mathbf{x}}\,(\mathbf{A}'_1\,\mathbf{A}_1 + \mathbf{B}'_1\,\mathbf{B}_1) \tag{6.1.11b}$$

The residual sum of squares $S'^2_e$ in this case is

$$\mathbf{y}'\,\mathbf{y} + \mathbf{x}'\,\mathbf{x} - \hat{\hat{\boldsymbol{\phi}}}'\,(\mathbf{A}'_1\,\mathbf{y} + \mathbf{B}'_1\,\mathbf{x}) \tag{6.1.12b}$$

Equating the derivative of (6.1.12b) with respect to $\mathbf{x}$ to zero, we get

$$2\mathbf{x} - \frac{d\bar{\boldsymbol{\phi}}'}{d\mathbf{x}}\,(\mathbf{A}'_1\,\mathbf{y} + \mathbf{B}'_1\,\mathbf{x}) - \mathbf{B}_1\,\hat{\hat{\boldsymbol{\phi}}} = 0 \tag{6.1.13b}$$

Remembering (6.1.11b), we finally get

$$\mathbf{x} = \mathbf{B}_1\,\hat{\hat{\boldsymbol{\phi}}} \tag{6.1.14b}$$

Substituting (6.1.14b) in (6.1.10b), we get the same normal equations as in (6.1.7a) and substituting (6.1.14b) in (6.1.12b) we get the same conditional residual sum of squares as in (6.1.8a). Hence

Sum of Squares due to hypothesis = Min (conditional Residual Sum of squares) − Min (Residual Sum of Squares)

The sum of squares due to hypothesis calculated from the completed table will, therefore, always have a positive bias.

In actual practice, when the analysis of the original design is known, much of the calculations can be simplified if we remember (6.1.8b). Thus in a random-ised block experiment with $t$ treatments and $r$ blocks where the expectation of the yield of the $j$th treatment in the $i$th block is $\mu + \alpha_i + \tau_j$, it is well known that the least square solutions for $\mu$, $\alpha_i$ and $\tau_j$ are respectively the grand mean, difference between $i$ th block mean and grand mean and difference between $j$ th treatment mean and grand mean. If, therefore, this observation is miss-ing, we represent by $x$, the missing yield, by $B_i$, the actual $i$th block total by $T_j$, the actual $j$th treatment total and by $G$, the actual grand total. By (6.1.8b)

$$x = \frac{G+x}{tr} + \left(\frac{B_i + x}{t} - \frac{G+x}{tr}\right) + \left(\frac{T_j + x}{r} - \frac{G+x}{tr}\right)$$

whence $x = \dfrac{r\,B_i + t\,T_j - G}{(r-1)\,(t-1)}$ (6.1.15)

If the yields of the $j$th treatment in the $i$th block and $j'$-th treatment in the $i'$-th block be missing and if $x$ and $w$ are substituted for the missing yields, we have if $i \neq i'$, $j \neq j'$

$$x = \frac{B_i + x}{t} + \frac{T_j + x}{r} - \frac{G + x + w}{tr}$$

$$w = \frac{B_{i'} + w}{t} + \frac{T_{j'} + w}{r} - \frac{G + x + w}{tr}$$

Whence we get easily

$$x = \frac{(r-1)(t-1)[rB_i + tT_j - G] - [rB_{i'} + tT_{j'} - G]}{(r-1)^2(t-1)^2 - 1}$$

$$w = \frac{(r-1)(t-1)[rB_{i'} + tT_{j'} - G] - [rB_i + tT_j - G]}{(r-1)^2(t-1)^2 - 1}$$

The estimate of the missing plot can also be obtained by utilising the method of covariance analysis. Let $x = 0$ and $y =$ the actual yield for the existing units and $x = -1$ and $y = 0$ for the missing unit. The best estimate of the missing yield is simply the error $b$ regression coefficient. Thus for the randomised block with one missing unit, we have the covariance analysis

| | $S(x^2)$ | $S(xy)$ | $b$ |
|---|---|---|---|
| Blocks | $\frac{1}{t} - \frac{1}{rt}$ | $-\frac{B_i}{t} + \frac{G}{rt}$ | |
| Treatments | $\frac{1}{r} - \frac{1}{rt}$ | $-\frac{T_j}{r} + \frac{G}{rt}$ | |
| Error | $(r-1)(t-1)/rt,$ | $\frac{B_i}{t} + \frac{T_j}{r} - \frac{G}{rt},$ | $\frac{rB_i + tT_j - G}{(r-1)(t-1)}$ |
| Total | $1 - \frac{1}{rt}$ | $\frac{G}{rt}$ | |

If the yield of the plot in the $i$th row and $j$th column containing the $k$th treatment in the $r \times r$ Latin Square be missing and if $R_i$, $C_j$, $T_k$ and $G$ denote respectively the actual row, column, treatment and grand totals we have $x$, the missing yield given by

$$x = \frac{R_i + x}{r} + \frac{C_j + x}{r} + \frac{T_k + x}{r} - \frac{2(G + x)}{r^2}$$

whence

$$x = \frac{r(R_i + C_j + T_k) - 2G}{(r-1)(r-2)}$$

The covariance technique would also give the same result :

| | $S(x^2)$ | $S(xy)$ | $b$ |
|---|---|---|---|
| Rows | $(1/r - 1/r^2)$ | $-R_i/r + G/r^2$ | ... |
| Columns | $(1/r - 1/r^2)$ | $-C_j/r + G/r^2$ | ... |
| Treatments | $(1/r - 1/r^2)$ | $-T_k/r + G/r^2$ | ... |
| Error | $(r-1)(r-2)/r^2,$ | $[r(R_i+C_j+T_k)-2G]/r^2,$ | $[r(R_i+C_j+T_k)-2G]/(r-1)(r-2)$ |
| Total | $1 - 1/r^2$ | $G/r^2$ | ..................... |

When performing the analysis of variance, it is also unnecessary to obtain the conditional error sum of squares. Thus if we obtain the actual block sum of squares and total sum of squares, then by subtracting from the total sum of squares, the block sum of squares and error sum of squares, we obtain the actual treatment sum of squares. In the randomised block experiment with one missing yield, we have seen that (6.1.15) provides an estimate of the missing value. Evidently,

$E(x) = \mu + \alpha_i + \tau_j$.

$$V(x) = \frac{r^2(t-1) + t^2(r-1) + rt - 1) - 2r(t-1) - 2t(r-1)}{(r-1)^2\,(t-1)^2}\,\sigma^2$$

$$= \frac{r+t-1}{(r-1)(t-1)}\,\sigma^2 \qquad (6.1.16)$$

The variance of the estimate of a difference between a treatment with one missing unit and a treatment without any missing unit $= V\left(\frac{T_j + x}{r} - \frac{T_{j'}}{r}\right)$

$$= \frac{\sigma^2}{r^2}\left[(r-1) + \frac{r+t-1}{(r-1)(t-1)} + r + \frac{2t(r-1) - 2r(t-1)}{(r-1)(t-1)}\right]$$

$$= \frac{2\sigma^2}{r}\left[1 + \frac{t}{2(r-1)\,(t-1)}\right] \qquad (6.1.17)$$

Whereas the variance between two treatment means without any missing unit is $\frac{2\sigma^2}{r}$.

Average variance of treatment differences

$$= \frac{\frac{2\sigma^2}{r}\left[1 + \frac{t}{2(r-1)(t-1)}\right](t-1) + \frac{2\sigma^2}{r}\,\frac{(t-1)(t-2)}{2}}{t(t-1)/2}$$

$$= \frac{2\sigma^2}{r}\left[1 + \frac{1}{(r-1)(t-1)}\right] \qquad (6.1.18)$$

Hence the loss of efficiency is $1/[(r-1)(t-1)+1]$.

The actual treatment sum of squares

$$= \Sigma y^2 - \left[x^2 + \Sigma y^2 - \frac{B^2_1 + B^2_2 + \ldots\ldots + (B_i + x)^2 + \ldots\ldots + B_r^2}{t}\right.$$
$$\left. - \frac{T^2_1 + T^2_2 + \ldots\ldots + (T_j + x)^2 + \ldots\ldots + T_t^2}{r} + \frac{(G+x)^2}{tr}\right]$$
$$- \frac{B^2_1 + \ldots\ldots + B^2_{i-1} + B^2_{i+1} + \ldots\ldots + B^2_r}{t} + B^2_i/(t-1)$$

$$= \left[\frac{T^2_1 + \ldots\ldots + (T_j + x)^2 + \ldots\ldots + T^2_t}{r} - \frac{(G+x)^2}{tr}\right] - \text{Bias}.$$

$$\therefore \text{Bias} = \frac{B^2_1}{t(t-1)} + \frac{x^2(t-1)}{t} - \frac{2B_i x}{t} = \frac{t-1}{t}\left[x - \frac{B_i}{t-1}\right]^2$$

The result is quite general. In fact, the sum of squares due to hypothesis in the analysis of variance table with substitutions for the missing yields will always be greater than the actual sum of squares due to hypothesis and if this turns out to be non-significant compared to the error sum of squares with *appropriate* degrees of freedom, the actual sum of squares due to hypothesis will also turn out to be non-significant.

2. *Fractional Replication.* In factorial experiments, when the number of factors is large or one or more of the factors include a large number of levels, we have a great number of treatment combinations. Thus with 6 factors each at three levels, we have as many as 729 treatments. In situations like this, the resources of the experimenter may not be sufficient to have even one replication. The device of fractional replication, if properly employed, is useful in such situations.

Let us consider the symmetrical factorial design $s^m$ where $s$ is a power of a prime. Let $P(a_{i1}, a_{i2}, \ldots, a_{im}),\ i = 1, 2, \ldots, k$ (6.2.1)

where each $a_{ij}$ is an element of $GF(s)$, denote $k$ independent pencils. Then in a $(1/s^k)$ replicate of the $s^m$ design, we shall take only the $s^{m-k}$ treatments satisfying

$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{im}x_m = \alpha_{j_i},\ j = 1, 2, \ldots, k$ where each $\alpha_{j_i}$ belongs to $GF(s)$. Usually each $\alpha_{j_i}$ on the right hand side is taken equal to $\alpha_0$, the null element. Thus of the $s^m$ treatments $(x_1, x_2, \ldots, x_m)$ we retain only the $s^{m-k}$ treatments satisfying

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{im}x_m = 0 \quad i = 1, 2, \ldots, k \tag{6.2.2}$$

Let $P(b_1, b_2, \ldots, b_m)$ be a pencil independent of (6.2.1). Let

$\sum_{\alpha\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_k}}$ denote the sum of treatments satisfying

$$b_1x_1 + b_2x_2 + \cdots + b_mx_m = \alpha$$

$$a_{j1}x_1 + a_{j2}x_2 + \ldots a_{jm}x_m = \alpha_{ij},\ j = 1, 2, \ldots, k$$

Then any contrast carried by the pencil $P(b_1, b_2, \ldots, b_m)$ may be written in the form

$$l_1 \underset{\alpha_{i_1}, \ldots, \alpha_{i_k}}{S} \sum 0\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k} + l_2 \underset{\alpha_{i_1}, \ldots, \alpha_{i_k}}{S} \sum \alpha_1, \alpha_{i_1}, \alpha_{i_2} \ldots, \alpha_{i_k} + \ldots l_s \underset{\alpha_{i_1}, \ldots, \alpha_{i_k}}{S} \sum \alpha_{s-1}, \alpha_{i_1}, \ldots, \alpha_{i_k} \tag{6.2.3}$$

where $l_1 + l_2 + \ldots + l_s = 0$ and $S$ stands for summation. In terms of the retained treatments, (6.2.3) can be written as

$$l_1 \ \Sigma_{0\,0\ldots\ldots0} + l_2 \ \Sigma\alpha_{1\,0\ldots\ldots0} + \cdots + l_s \ \Sigma\alpha_{s-1}\,0\ldots.0 \tag{6.2.4}$$

The contrasts carried by any one of the $s^k$ pencils

$$P\left\{ b_1 + \sum_{i=1}^{k} \lambda_i a_{i1}, \ldots, b_m + \sum_{i=1}^{k} \lambda_i a_{im} \right\} \text{ are}$$

$$\begin{aligned} &l_1 \quad S \\ &\alpha_{i_1}, \ldots, \alpha_{i_k} \sum \alpha_0 - \lambda_1 \alpha_{i_1} - \ldots - \lambda_k \alpha_{i_k} + \ldots + \\ &l_s \quad S \\ &\alpha_{i_1}, \ldots, \alpha_{i_k} \sum \alpha_{s-1} - \lambda_1 \alpha_{i_1} \ \ldots - \lambda_k \alpha_{i_k}, \alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k} \end{aligned} \tag{6.2.5}$$

and they become in terms of the retained treatments exactly (6.2.4). Summing up (6.2.5) over all values of $\lambda_1, \lambda_2, \ldots, \lambda_k$ we get

$$\begin{aligned} &s^k [l_1 \Sigma_{00\ldots0} + l_2 \Sigma \alpha_{10\ldots0} + \ldots + l_s \Sigma \alpha_{s-1} 0 \ldots 0 \\ + \quad &s^{k-1} \{l_1 [\Sigma_0 + \Sigma\alpha_1 + \ldots + \Sigma\alpha_{s-1} - (\Sigma 00 \ldots 0 + \ldots + \Sigma\alpha_{s-1} 00 \ldots 0)] \\ &+ \quad l_2 [ \qquad\qquad\qquad\qquad ] + \ldots\ldots \qquad\qquad \} \\ &= s^k [ l_1 \Sigma_{0\,0\ldots0} + l_2 \Sigma\alpha_{1\,0\ldots0} + \ldots + l_s \Sigma \alpha_{s-1\,0\ldots0} \ ] \\ &+ s^{k-1} ( l_1 + l_2 + \ldots + l_s ) [ \qquad\qquad\qquad\qquad ] \\ &= s^k [ l_1 \Sigma_{00\ldots0} + l_2 \Sigma \alpha_1 00\ldots0 + \ldots + l_s \Sigma \alpha_{s-1} 0\ldots0 ] \end{aligned}$$

Thus each pencil is one of a set of $s^k$ alias set of pencils such that the sum of similar contrasts carried by all of them is estimable. The result of using a fractional replication is that the contrasts are no longer separately estimable, but the sum of the contrasts belonging to any complete alias set is estimable. If all the contrasts of any alias set with the exception of one contrast are of sufficiently high order to be negligible, then this particular contrast is estimable.

It is also evident that the sum of the contrasts carried by different alias sets are orthogonal. The total number of ways of getting pencils of the type (6.2.1) is the number of $(m\text{-}k-1)$-flats in the $(m-1)$-flat $x_0=0$ in $PG(m, s)$ and is equal to $\phi\,(m-1, m-k-1, s) = \phi\,(m-1, k-1, s)$ which wirtten out in full gives

$$\frac{(s^m - 1)(s^{m-1} - 1)\ldots(s^{m-k+1} - 1)}{(s^k - 1)(s^{k-1} - 1)\ldots(s - 1)}$$

In an $s^m$ design in $1/s^k$ replicate, it is sometimes necessary to resort to confounding for effective elimination of soil heterogeneity. The process is quite straightforward. We choose $g$ independent pencils

$$p\,(c_{i'1}, c_{i'2}, \ldots, c_{i'm}),\ i' = 1, 2, \ldots, g \tag{6.2.6}$$

which are also independent of (6.2.1) and assign to the $s^g$ blocks the treatments

$$\sum_{j=1}^{m} a_{ij}\, x_j = 0,\ i = 1, 2, \ldots, k$$

$$\sum_{j=1}^{m} c_{i'j}\, x_j = \alpha_{l_{i'}},\ i' = 1, 2, \ldots, g$$

The confounded effects belong to $s^g$ pencils and their alias sets. The sum of the contrasts carried by any pencil independent of these $k + g$ pencils and its aliases, is estimable. Let us illustrate this by the following example.

TABLE

*Half Replicate of a $2^6$ Experiment with Confounding to Reduce Block Size to Eight*

| $x_1$ | $x_2$ | $x_3$ | Block I | | | Block II | | | Block III | | | Block IV | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $x_4$ | $x_5$ | $x_6$ | $x_4$ | $x_5$ | $x_6$ | $x_4$ | $x_5$ | $x_6$ | $x_4$ | $x_5$ | $x_6$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 |
| 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |
| 0 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 |
| 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 |

This is an example of a half replicate of a $2^6$ design involving six factors $A, B, C, D, E$ and $F$ each at two levels 0 and 1 respectively. The retained treatments are $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \doteq 0$. Confounding was further resorted to for reducing block size, the confounded pencils being $P(0, 0, 1, 0, 0, 1)$ and $P(0, 1, 1, 0, 1, 0)$. Thus $CF = ABDE, BCE = ADF$ and $BEF = ACD$ are confounded with the blocks. The alias of a main effect is a five factor interaction and that of a first order interaction is a four-factor interaction.

3. *Confounded Asymmetrical Factorial Designs.* In the general case of a factorial design, the $m$ factors $F_1, F_2, \ldots, F_m$ usually do not all have the same number of levels. In such a case the experiment is known as asymmetrical. Problems of reducing block size by confounding only high order interactions do not always lead to solutions and even in favourable situations, the problem of construction of such designs is more difficult than the symmetrical case. Thus in the case of $s_1 \times s_2$ design, except in special cases, it is not possible to arrange the $s_1 \times s_2$ treatments in $s_2$ blocks of $s_1$ treatments each, without confounding some contrast belonging to the main effect of the second factor. Hence we try to have a number of replications such that the loss of information on every degree of freedom belonging to main effect of the second factor is uniform and the loss of information on every degree of freedom belonging to the two-factor interaction is also uniform. This leads to a balanced arrangement. A few special cases are described below :

(*i*) The case $2 \times s_2$. The arrangement in Table 6.1

TABLE 6·1

| Block | 1 | 2 | ... | $s_2$ | $s_2+1$ | $s_2+2$ | ... | $2s_2$ | ... | $(s_2-1)s_2+1$, | ... | $(s_2-1)s_2+2$, | ... | $s_2^2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Level of $F_1$ | Level of $F_2$ | | | | | | | | | | | | | |
| 1 | 1 | 1 | ... | 1 | 2 | 2 | ... | 2 | ... | $s_2$ | ... | $s_2$ | ... | $s_2$ |
| 2 | 1 | 2 | ... | $s_2$ | 1 | 2 | ... | $s_2$ | ... | 1 | ... | 2 | ... | $s_2$ |

TABLE 6·2

| Block | 1 | 2 | ... | $s_2$ | $s_2+1$ | $s_2+2$ | ... | $2s_2$ | ... | $(s_2-1)s_2+1$, | $(s_2-1)s_2+2$, | ... | $s_2^2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Level of $F_1$ | Level of $F_2$ | | | | | | | | | | | | |
| 1 | 1 | 2 | ... | $s_2$ | 1 | 2 | ... | $s_2$ | ... | 1 | 2 | ... | $s_2$ |
| 2 | 1 | 1 | ... | 1 | 2 | 2 | ... | 2 | ... | $s_2$ | $s_2$ | ... | $s_2$ |
| 3 | $\alpha_{11}$ | $\alpha_{21}$, | ... | $\alpha_{s_21}$ | $\alpha_{12}$ | $\alpha_{22}$ | ... | $\alpha_{s_22}$ | ... | $\alpha_{1s_2}$ | $\alpha_{2s_2}$ | ... | $\alpha_{s_2s_2}$ |

TABLE 6·3

| Block | 1 | 2 | ... | $s_2$ | $s_2+1$ | $s_2+2$ | ... | $2s_2$ | ... | $(s_2-1)s_2+1$ | ... | $(s_2-1)s_2+2$ | ... | $s_2^2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Level of $F_1$ | Level of $F_2$ | | | | | | | | | | | | | |
| 1 | 1 | 2 | ... | $s_2$ | 1 | 2 | ... | $s_2$ | ... | 1 | ... | 2 | ... | $s_2$ |
| 2 | 1 | 1 | ... | 1 | 2 | 2 | ... | 2 | ... | $s_2$ | ... | $s_2$ | ... | $s_2$ |
| 3 | 1 | $\alpha_{21}^{(1)}$ | ... | $\alpha_{s_21}^{(1)}$ | 2 | $\alpha_{22}^{(1)}$ | ... | $\alpha_{s_22}^{(1)}$ | ... | $s_2$ | ... | $\alpha_{2s_2}^{(1)}$ | ... | $\alpha_{s_2s_2}^{(1)}$ |
| ........ | ........ | ........ | ........ | ........ | ... | ........ | ........ | ........ | ... | ... | ........ | ........ | ........ | ........ |
| $s_1$ | 1 | $\alpha_{21}^{(s_1-2)}$ | ... | $\alpha_{s_21}^{(s_1-2)}$ | 2 | $\alpha_{22}^{(s_1-2)}$ | ... | $\alpha_{s_22}^{(s_1-2)}$ | ... | $s_2$ | | $\alpha_{2s_2}^{(s_1-2)}$ | ... | $\alpha_{s_2s_2}^{(s_1-2)}$ |

will lead to a confounded arrangement having parameters $v = 2\times s_2$, $b = s_2^2$, $k = 2$, $r = s_2$. Here the main effect of $F_1$ is unconfounded in each of the $s_2$ replications and there is a uniform loss of information $1/s_1$ on every degree of freedom belonging to either $F_2$ or $F_1 F_2$.

(*ii*) The case $3\times s_2, s_2 > 3$. A Latin Square of side $s_2$ can be constructed with $(1, 2, 3, \ldots, s_2)$ in the leading diagonal. Let

$$\begin{array}{cccc} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1s_2} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2s_2} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{s_2 1} & \alpha_{s_2 2} & \cdots & \alpha_{s_2 s_2} \end{array}$$

be such a Latin Square. Then the arrangement in Table 6·2

will lead to a confounded arrangement having parameters $v = 3s_2$, $b = s_2^2$, $k = 3$, $r = s_2$. Here the main effects of $F_1$ are preserved in each of the $s_2$ replications but there is a uniform loss of $1/s_1$ on every degree of freedom belonging to $F_2$ and $F_1 F_2$.

(*iii*) $v = s_1 \times s_2$ where $s_2$ is a power of a prime and $s_1 \leqslant s_2 + 1$.

In this case we can make use of orthogonal Latin Squares whose first rows are $1, 2, \ldots, s_2$ for constructing confounded arrangements with $v = s_1 \times s_2$, $b = s_2^2$, $k = s_1$, $r = s_2$, preserving the main effect of $F_1$ but with uniform loss of information $1/s_1$ on every degree of freedom belonging to $F_2$ and $F_1 F_2$.

| 1st Square | | | 2nd Square | | | $(s_1 - 2)$th Square | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 2……… | $s_2$ | 1 | 2……… | $s_2$ | 1 | 2 | $s_2$ |
| $\alpha_{21}^{(1)}$ | $\alpha_{22}^{(1)}$ …… | $\alpha_{2s_2}^{(1)}$ | $\alpha_{21}^{(2)}$ | $\alpha_{12}^{(2)}$ …… | $\alpha_{2s_2}^{(2)}$ | $\alpha_{21}^{(s_1-2)}$ | $\alpha_{22}^{(s_1-2)}$ | $\alpha_{2s_2}^{(s_1-2)}$ |
| ……… | ……… | ……… | ……… | ……… | ……… | ……… | ……… | ……… |
| $\alpha_{s_2 1}^{(1)}$ …… | …… | $\alpha_{s_2 s_2}^{(1)}$ | $\alpha_{s_2 1}^{(2)}$ | $\alpha_{s_2 2}^{(2)}$ …… | $\alpha_{s_2 s_2}^{(2)}$ | $\alpha_{s_2 1}^{(s_1-2)}$ …… | …… | $\alpha_{s_2 s_2}^{(s_1-2)}$ |

The balanced arrangement would be as given in Table 6·3

In the $s_1 \times s_2 \times \cdots, \times s_m$ design, if $s_{t+1} = s_{t+2} = \ldots = s_m = s$ (say) we can write it as $s_1 \times s_2 \times \ldots \times s_t \times s^{m-t}$. The $s^{m-t}$ combinations of the last

$(m-t)$ factors can be divided into $s^l$ groups of $s^{m-l-t}$ each. We can imagin these $s^l$ groups as the $s^l$ levels of a fictitious factor $F'_{t+1}$. By arranging the division of the $s^{m-t}$ sets into $s^l$ groups of $s^{m-l-t}$ each such that there is uniform sacrifice of information on interactions of the factors $F_{t+1}, F_{t+2}, \ldots, F_m$ we can obtain balanced confounded arrangements in the $s_1 \times s_2 \times \ldots \times s_t \times s^{m-t}$ design. For example in the $2 \times 3^2$ design, each of the following division of the $3^2$ design involving $F_2$ and $F_3$.

| I | II |
|---|---|
| (1, 2), (2, 1), (3, 3) | (1, 1), (2, 2), (3, 3) |
| (2, 2), (3, 1), (1, 3) | (3, 2), (2, 1), (1, 3) |
| (3, 2), (1, 1), (2, 3) | (1, 2), (3, 1), (2, 3) |

into 3 sets of 3 each will confound 2 d.f. belonging to $F_2F_3$. Associating thes sets with the levels of a fictitious factor $F'_2$, we can from case $(i)$ of section 2, obtain the design given the Table 6·4

TABLE 6·4

| Blocks | 1 2 3 | 4 5 6 | 7 8 9 | 10 11 12 | 13 14 15 | 16 17 18 |
|---|---|---|---|---|---|---|
| Levels of $F_2\,F_1$ | | | Level of $F_3$ | | | |
| 2 1 | 1 1 1 | 2 2 2 | 3 3 3 | 2 2 2 | 1 1 1 | 3 3 3 |
| 1 1 | 2 2 2 | 3 3 3 | 1 1 1 | 1 1 1 | 3 3 3 | 2 2 2 |
| 3 1 | 3 3 3 | 1 1 1 | 2 2 2 | 3 3 3 | 2 2 2 | 1 1 1 |
| 2 2 | 1 2 3 | 1 2 3 | 1 2 3 | 2 1 3 | 2 1 3 | 2 1 3 |
| 1 2 | 2 3 1 | 2 3 1 | 2 3 1 | 1 3 2 | 1 3 2 | 1 3 2 |
| 3 2 | 3 1 2 | 3 1 2 | 3 1 2 | 3 2 1 | 3 2 1 | 3 2 1 |

Omitting the blocks, 1,5, 9, 10, 14 and 18 we get another good design with $F_2 F_3$ and $F_1 F_2 F_3$ partially confounded. Many papers have appeared in recent years on construction of balanced confounded designs in the asymmetrical case but a fuller discussion is not possible in a book of this size.

4. *Weighing Designs.* When we want to find out the weights of a number of light objects, it is possible to increase the precision of weighings by taking a suitable combination of the objects instead of weighing them separately which is the usual practice. Let $\sigma^2$ be the variance of a single weighing. Then weighed separately, each of two objects will be estimated with variance $\sigma^2$. If however, both objects are placed on one pan and the estimated weight is $y_1$ and the two objects are placed on different pans and $y_2$ is the estimated weight placed on the pan containing the lighter of the two objects to obtain balance, then the estimates of the weights of the two objetcs are $(y_1 + y_2)/2$ and $(y_1 - y_2)/2$ and the variance of each estimate is $\sigma^2/2$. Evidently the second procedure is superior to the ordinary method of weighing the two objects separately.

Let $x_{i\alpha}$ take the value 1, if, the $i$th object be included in the $\alpha$th weighing in the right pan, the value $-1$ if the $i$th object be included in the $\alpha$th weighing in the left pan and the value 0 if it is not included in the $\alpha$th weighing. Let there be $p$ objects having true weights $\beta_1, \beta_2, \ldots, \beta_p$ and let there be $N$ weighings, the observed results being $y_1, y_2, \ldots, y_N$. Now

$$E(\mathbf{y}) = E\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} x_{11} & x_{21} \ldots x_{p1} \\ x_{12} & x_{22} \ldots x_{p2} \\ \ldots & \ldots \\ x_{1N} & x_{N} \ldots x_{pN} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_a \\ \ldots \\ \beta_p \end{bmatrix} = \mathbf{X}\,\boldsymbol{\beta} \ldots \qquad (6.4.1)$$

In order to find estimates of $\boldsymbol{\beta}$ the usual method is to obtain those $\boldsymbol{\beta}$ which will minimise

$$(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) = \mathbf{y}'\mathbf{y} - 2\,\boldsymbol{\beta}'\,\mathbf{X}'\mathbf{y} + \boldsymbol{\beta}'\,\mathbf{X}'\mathbf{X}\boldsymbol{\beta} \qquad (6.4.2)$$

Equating the derivative of (6.4.2) with respect to $\boldsymbol{\beta}$ to zero, we have estimates $\hat{\boldsymbol{\beta}}$ of $\boldsymbol{\beta}$ given by

$$\mathbf{X}'\mathbf{y} = \mathbf{X}'\,\mathbf{X}\,\hat{\boldsymbol{\beta}} \qquad (6.4.3)$$

Assuming $\mathbf{x}'\mathbf{x}$ to be non-singular, we can write $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\,\mathbf{y}$. Evidently $E(\hat{\boldsymbol{\beta}}) = \boldsymbol{\beta}$, $E(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})' = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}$. Let $C = \det(\mathbf{X}'\mathbf{X}) = |c_{ij}|$ $i, j = 1, 2, \ldots, p$ and let $C_{ij}$ denote the co-factor of $c_{ij}$ in $C$. Further, we denote $(\mathbf{x}'\mathbf{x})^{-1}$ by $(c^{ij})$. Now

$$C = c_{11}\,C_{11} - \sum_{i,j=2}^{p} C_{ij}\,c_{1i}\,c_{1j}$$

$$c^{11} = \frac{1}{c_{11}} + \frac{1}{c_{11}\,C} \sum_{i,j=2}^{p} C_{ij}\;c_{1i}\;c_{1j}$$

The second constituent on the right hand side is a positive definite quadratic form; hence the minimum value of $c^{11}$ will occur when $c_{12} = c_{13} = \ldots, = c_{1p} = 0$ and the absolute minimum value $1/N$ will occur when further $x_{1j}^2 = \pm 1, j = 1, 2, \ldots, N$. Thus when $\mathbf{X}'\mathbf{X} = N\,\mathbf{I}_p$, all the objects will be weighed with maximum precision.

This introduces the desirability of constructing $N \times p$ Hadamard matrices $\mathbf{X}$ with elements $+1$ or $-1$ such that $\mathbf{X}'\,\mathbf{X} = N\,\mathbf{I}_p$. Since

$$\begin{pmatrix} \mathrm{A} & \mathrm{A} \\ \mathrm{A} & -\mathrm{A} \end{pmatrix}' \begin{pmatrix} \mathrm{A} & \mathrm{A} \\ \mathrm{A} & -\mathrm{A} \end{pmatrix} = \begin{pmatrix} \mathrm{A}' & \mathrm{A}' \\ \mathrm{A}' & -\mathrm{A}' \end{pmatrix} \begin{pmatrix} \mathrm{A} & \mathrm{A} \\ \mathrm{A} & -\mathrm{A} \end{pmatrix} = \begin{pmatrix} 2\mathrm{A}'\mathrm{A} & \mathrm{O} \\ \mathrm{O} & 2\mathrm{A}'\mathrm{A} \end{pmatrix}$$

and $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is a Hadamard matrix, a Hadamard matrix of order $2^k$ can always be constructed.

In weighings with a spring balance, each $x_{ij}$ can have the values 1 or 0. The incidence matrix $\mathbf{N}$ of a design in which a treatment can occur at most once in a block may be employed to construct a weighing design with a spring balance by identifying objects with varieties, blocks with weighings and $\mathbf{N}'$ with $\mathbf{X}$. Thus in the case of a *BIBD*

$$(\mathbf{X}'\,\mathbf{X})^{-1} = (\mathbf{N}\,\mathbf{N}')^{-1} = \frac{1}{r-\lambda}\,\mathbf{I}_v - \frac{\lambda}{(r-\lambda)\,rk}\,\mathbf{E}_{vv}$$

Hence the variance of the estimated weight of each of the $v$ objects is

$$\sigma^2\,(rk - \lambda)\,/\,(r-\lambda)\,rk$$

The efficiency of a weighing design is often defined as $\dfrac{\text{Min det }(\mathbf{X}'\,\mathbf{X})^{-1}}{\det\,(\mathbf{X}'\,\mathbf{X})^{-1}} = \dfrac{\det\,(\mathbf{X}'\,\mathbf{X})}{\text{Max det }(\mathbf{X}'\,\mathbf{X})}$. When a Hadamard matrix $H_{N+1}$, of order $(N+1)$ exists, a spring balance weighing design of maximum efficiency involving $N$ weighings of $N$ objects can be easily constructed. We have merely to "sweep out" the 1st column of $H_{N+1}$, omit the first row and 1st column and replace the non-zero elements of the resulting matrix of order $N$ by $+1$. The determinant of such a matrix is $(N+1)^{\frac{N+1}{2}} \big/ 2^N$ in absolute value. According to this definition of efficiency

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

is the best weighing arrangement with three weighings of three objects in a spring balance. The arrangement can also be identified with the retained treatments in a half replicate of a $2^3$ design: $x_1 + x_2 + x_3 = 0$, barring the combination 0 0 0.

5. In section 6 of Chapter 3, we have shown that a symmetrical BIBD can always be converted into a Youden Square. Another method which is more helpful in actual construction is to write down the symmetrical BIBD as follows :

| | Blocks |
|---|---|
| Positions | 1 2........................$b$ |
| 1 | |
| 2 | |
| . | |
| . | |
| $k$ | |

As a measure of " non-Youdenness " we shall take $M = \sum_{i,j} M(i, j)$, where $M(i, j)$ is 0 if the $j$th variety occurs in the $i$th position and 1 otherwise, $i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, v$. For a Youden Square $M = 0$. If $M > 0$, a particular variety, say $j$, will not occur in the $i$th position and will occur twice in some other position $i'$. The situation is represented below

| | Block | |
|---|---|---|
| Position | $x$ | $y$ |
| $i$ | $j_1$ | $j_2$ |
| $i'$ | $j$ | $j$ |

where both treatments $j_1$ and $j_2$ are different from $j$. Interchange of $j_2$ and $j$ in block $y$ will normally lead to a reduction in the value of $M$ except when $j_2$ already occurred in position $i'$ and $j_2$ did not occur in position $i$ except the one place indicated. In this case, after the interchange, $j_2$ occurs in two places in position $i'$ and is absent in position $i$. We may repeat the procedure with $j_2$ and it can be shown that ultimately $M$ will be reduced by at least 1. Thus by repeated applications of this system of interchanges, we can reduce $M$ to zero.

By a slight modification of the above arguments, one can prove easily that a balanced incomplete block design, in which the number of blocks is $v\,m$, an integral multiple of $v$, the number of treatments, can be converted into a design in which every treatment will occur $m$ times exactly in each of the $k$ positions. In the notations of section 2 of chapter 2, $\mathbf{L} = m\,\mathbf{E}_{vk'}$, $\mathbf{NN}' = (r - \lambda)\,\mathbf{I}_v + \lambda\,\mathbf{E}_{vv}$. Hence $\mathbf{F}$ of (2.2.4) is $\frac{\lambda v}{k}\,\mathbf{I}_v - \frac{\lambda}{k}\,\mathbf{E}_{vv}$, the eigen values of which are 0 and $\frac{v\lambda}{k}$ having multiplicities 1 and $v - 1$ respectively. Hence,

all elementary treatment contrast will be estimated with variance $\dfrac{2k\sigma^2}{v\lambda}$. Further (2.2.11) is satisfied and thus the estimates of row contrasts are orthogonal to estimates of treatment contrasts.

## EXERCISES TO CHAPTER VI

1. In the following $5 \times 5$ Latin Square Experiment, certain plot yields marked with an asterisk are missing

| | | | | |
|---|---|---|---|---|
| A | * | C | D | E |
| B | C | D | E | A |
| C | D | * | A | * |
| D | E | A | B | C |
| E | A | B | C | D |

Obtain the variances of different treatment comparisons and calculate the loss of efficiency due to the non-availability of yields of the three plots.

2. Give the analysis appropriate to an $s \times s$ Latin Square when a single row, column or treatment is missing.

3. In a balanced incomplete block design, calculate estimates of different treatment comparisons with their variances when (*i*) the yield of one plot is missing and (*ii*) the yields of all plots in one particular block are missing. Calculate the loss of efficiency in each case.

4. In a split-plot experiment arranged in $r$ randomised blocks, having $\alpha$ levels of main treatments $A$ and $\beta$ levels of subtreatments $B$, the yield of a subplot is missing. Calculate standard error of the estimate of (*i*) difference between two $A$ means (*ii*) difference between two $B$ means (*iii*) difference between two $B$ means at the same level of $A$ and (*iv*) difference between two $A$ means at the same level of $B$.

5. In a randomised block experiment, the yields of treatments 1 and 2 in block 1 have got mixed up through accident, so that although the individual plot yields are not separately available, the total yield of the two plots is known. Obtain the best unbiased linear estimates of the separate yields. Compute the variances of different treatment comparisons and find the loss of efficiency due to the mixing up of the plots.

6. In the following $4 \times 4$ Latin Squares experiment, the yields of the two plots marked with asterisk got mixed up

| | | | |
|---|---|---|---|
| A | B | C | D |
| C | * | D | B |
| * | C | B | A |
| B | D | A | C |

Obtain best unbiased linear estimates of the separate yields. Obtain formulae for the estimates when the mixed up plots belong to the same row, column or treatment.

7. Below is given an one-quarter replicate of a $2^8$ design where confounding has also been resorted to for reducing block size :

Block I: *ab, acf, beg, aeh, cefg, cd, abcdgh, bdf, abdefg, abcefh, gh, bcfgh, adfgh, acdeg, defh, bedch.*

Block II: *bcf, aeg,* (1), *beh, abcefg, abcd, cdgh, adf, defg, cefh, abgh, acfgh, bdfgh, bcdeg, abdefh, acdeh.*

Block III: *ce, bef, acg, bch, abfg, abde, degh, acdef, cdfg, fh, abcegh, aefgh, bcdefgh, bdg, abcdfh, bdh.*

Block IV: *de, bcdef, adg, bdh, abcdfg, abce, cegh, aef, fg, cdfh, abdegh, acdefgh, befgh, beg, abfh, ach.*

Identify the alias sub-group of pencils and the confounded interactions. Write down the aliases of the main effects and two-factor interactions. Assuming interactions of 3 or more factors to be negligible, give the partitioning of the total degrees of freedom in the analysis of variance table.

8. In a cotton mill, one of 5 spindles was found to be winding defective weft. In detecting the cause of the defect, the four component parts of the spindle designated by the letters $A, B, C, D$ are interchanged. The reconstructed spindles are then tested five at time over five periods $P$. You have the technologist's assurance that there is no interaction between the factors $P, A, B, C$ and $D$. Design a $1/5^3$ replicate of a $5^5$ design choosing a suitable alias sub-group for testing the effects of the various factors.

9. Construct a $4^5$ design in $1/4$ replicate with confounding to reduce the size of the block to 16 plots in which only 3 degrees of freedom from one first order interaction are confounded. Give the constitution of the principal block and the partition of the total degrees of freedom in the analysis of variance.

10. Show that if the alias subgroup involves only interactions of the $(t+k-1)$-th and higher orders $(t>k)$, the alias of an interaction of order $(k-1)$ would be $(t-1)$-th and higher order interactions. Hence or otherwise show that if an alias subgroup involves interactions of only the fourth and higher orders, the main effects and first order interactions are estimable, when interactions of second and higher orders are negligible.

11. Show that if we adopt a hypercube $(m, s, t, d)$ of strength $d$ as $\frac{1}{s^{m-t}}$ replicate of a $s^m$ design, interactions upto order $k-1$ where $2k < d+1$ are estimable when interactions of order equal to or greater than $d-k$ are negligible.

12. Six factors are available each at three levels. Show that it is possible to arrange the experiment so that (*i*) it is in 9 blocks of 27 plots each and forms (1/3) replicate of the $3^6$ design (*ii*) the aliases of main effects and two-factor interactions are interactions of at least four factors and (*iii*) two of the degrees of freedom between blocks may be interpreted as an interaction of not less than three factors, while the remaining six degrees of freedom between blocks are interactions of at least four factors.

Write down the constitution of the key block and indicate how others may be found. If residual error is estimated from four-factor interactions, ascertain the number of degrees of freedom that will be available for the purpose.

13. Construct a confounded $q \times 2^2$ design in blocks of $q \times 2$ plots in $q$ replications such that the loss of information on $B\,C$ is $(q-2)^2/q^2$ and in $(q-1)$ d.f. belonging to $A\,B\,C$ is $4(q-1)/q^2$. Indicate how the design can be generalised so as to give $q \times 2^n$ designs in blocks of $q \times 2^{n-1}$.

14. Plan a confounded $q \times p^2$ design where $p$ is either an odd prime or a power thereof in blocks of $q \times p$ plots in $q(p-1)/2$ replications such that the loss of information on the $(p-1)$ degrees of freedom belonging to $B\,C$ is $[(p-1)\ q^2 - 2pq + 2q]/q^2$ and that on the $(q-1)\ (p-1)$ degrees of freedom for $A\,B\,C$ is $2p(q-1)/q^2$. Indicate how this design can be extended to $q \times p^n$ design in blocks of $q \times p^{n-1}$ plots.

15. Let $p$ be a prime or power of a prime and $q$ be a positive integer less than $p$. Write the $p-1$ orthogonal Latin Squares of side $p$ and let the row number denote level of the first factor. If we omit from the squares the numbers $q+1, q+2, \ldots, p$ and suppose the remaining numbers indicate levels of the second factor, we get a $p \times q$ confounded design in blocks of $q$ plots in $(p-1)$ replications. Show that the losses of information on $A$, $B$ and $AB$ are respectively $(p-q)/q(p-1)$, 0 and $p/q(p-1)$.

16. In Ex. 15, suppose $q = p$ and we take $k$ replications where $1 \leqslant k \leqslant (p-1)$. Show that in this case only interaction $AB$ is confounded with blocks. Calculate the loss of information on $AB$.

17. Use Ex. 15 to construct a $p \times r \times s$ confounded design where $p$ has the same meaning as in Ex. 15 and $r \times s = q\, p$ by the substitution of the $r \times s$ combinations for the $q$ levels in that example. Illustrate your method by constructing the $4 \times 2 \times 2$, $5 \times 2 \times 2$ designs in blocks of 4 plots or $7 \times 3 \times 2$ design in blocks of 6 plots. Show that the loss of information on $A$ is $(p - rs)/rs\,(p - 1)$ and that on each of $AB$, $AC$, $ABC$ is $p/rs\,(p - 1)$.

18. Given $n$ sets $(a_1, a_2, \ldots, a_n)$ where each $a_i$ can assume $q$ values $0, 1, 2, \ldots, q - 1$. A subset $N$ of these sets is called an orthogonal array of strength 2 if all combinations of every pair of coordinates occur an equal number of $p = N/q^2$ times. Show that these orthogonal arrays can be utilised to construct designs for $s_1 \times s_2$ experiment in blocks of $s_1$ or $s_2$ plots preserving the main effects of $A$ or $B$ respectively.

19. Show that the average variance of the estimates of $p$ objects in $N$ weighings is $(\sigma^2/p)$. Trace $(\mathbf{X}'\,\mathbf{X})^{-1}$ and in the most efficient design it is $\sigma^2/N$. Hence the efficiency of a weighing design can be taken as

$p/N$ Trace $(\mathbf{X}'\,\mathbf{X})^{-1} = (1/N)$ Harmonic mean of the eigen values of $(\mathbf{X}'\,\mathbf{X})$
Obtain the efficiencies of the following weighing designs :

$$(1)\ \mathbf{X} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix} \qquad (2)\ \mathbf{X} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & -1 \\ -1 & -1 \end{bmatrix} \qquad (3)\ \mathbf{X} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

20. If $\mathbf{H}_{N-1}$ is a matrix of $(N - 1)$ rows and $p$ columns consisting of $+1$ and $-1$ such that $\mathbf{H}'_{N-1}\,\mathbf{H}_{N-1} = (N - 1)\mathbf{I}_p$, show that the efficiency of the weighing design

$$\mathbf{X} = \begin{bmatrix} \mathbf{E}_{1p} \\ \mathbf{H}_{N-1} \end{bmatrix}$$

is $(N - 1)\,(N + p - 1)/\,N(N + p - 2)$. Obtain the efficiency when

$$(i)\quad \mathbf{X} = \begin{bmatrix} \mathbf{E}_{rp} \\ \mathbf{H}_{N-1} \end{bmatrix}$$

$$(ii)\quad \mathbf{X} = \begin{bmatrix} \mathbf{E}_{r1}\mathbf{O} \\ \mathbf{H}_{N-1} \end{bmatrix}$$

21. Show that the most efficient weighing design for $N = 2^m$ can be written down by writing (1) $a$, $b$, ..., $ab$, ..., $abc$, along columns and $A$, $B$, ..., $AB$, ..., $ABC$... along rows and filling up the body of the table by 1 or $-1$

according as the rows have an even or odd number of letters in common with the columns and finally adjoining a row of + 1's.

22. If $N=2\,(p^n+1)=4\,k$, where $p$ is an odd prime or zero, show that an $N \times N$ matrix A can be constructed with + 1 and $-1$ such that $\mathbf{A}'\,A=N\mathbf{I}_N$.

23. We are interested in the total weight of $v$ objects by means of a spring balance. Naturally all the objects can not be weighed simultaneously. We take as the weighing design $\mathbf{X}=\mathbf{N}'$ where $\mathbf{N}$ is the incidence matrix of a BIBD with parameters $v, b, r, k, \lambda$. Show that the variance of the best unbiased linear estimate of the total weight is $\sigma^2 v/rk$.

24. A weighing design has the following **X** and **y** matrices :

$$\mathbf{X}=\begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix} \quad \mathbf{y}=\begin{bmatrix} 20 \\ 30 \\ -10 \\ -5 \\ -1 \end{bmatrix}$$

(*i*) Obtain the best estimates of individual weights

(*ii*) Calculate the efficiency of the design

(*iii*) Indicate how you would proceed if you feel that there is bias in the balance.

(*iv*) Can you estimate the bias from the above material? If not, state what more information you need.

25. A square matrix with a single entry of 1 in each row and in each column and with all other entries zero is called a permutation matrix. Show that the incidence matrix of a symmetrical BIBD with $k$ plots per block can be expressed as a sum of $k$ permutation matrices.

26. A certain dance is attended by $n$ boys and $n$ girls. Each boy has been previously introduced to exactly $k$ girls and each girl has been previously introduced to $k$ boys. No one wants to make any further introduction. Under these assumptions, will it be always possible for the boys and girls to be paired with each other in such a way that no further introductions are necessary?

27. An $r \times s$ Latin rectangle is an array of $r$ rows and $s$ columns formed from the integers 1, 2, ..., $n$ such that the integers in each row and in each column are distinct. It is said to be extendable to an $n \times n$ Latin Square if it is possible to adjoin $n-s$ columns and $n-r$ rows in such a way that the resulting array is an $n \times n$ Lain Square. Let $N(i)$)denote the number of times

the integer $i$ occurs in the Latin Rectangle. Show that a necessary and sufficient condition in order that the Latin Rectangle may be extended to an $n \times n$ Latin Square is that $N(i) \leqslant r + s - n$, $i = 1, 2, \ldots, n$.

28. Suppose that we are given $vr$ elements made up of $v$ varieties of objects each repeated $r$ times and that the set is arbitrarily arranged in a two-way classification of $r$ rows and $v$ columns. Show that it is always possible to rearrange the elements in each column so that each row will contain one and only one of each variety.

29. A pack of playing cards contain the numbers Ace, King, Queen, Jack, 10, 9, 8, ..., 2 of four suits. Arrange the 52 cards into 13 tricks of 4 cards each in such a way that (*i*) any two tricks have one number in common (*ii*) any two numbers occur together in some trick and (*iii*) any trick contains all four suits.

30. Thirty-one plants chosen for an experiment have each six leaves growing serially along the stem. Plan an allocation of the 31 treatments to the leaves such that each pair of treatments occurs once on the same plant and each treatment occurs once on each leaf position.

# LIST OF REFERENCES

Below are listed the principal sources consulted by the author while writing the present book.

## I. BOOKS


1. BOSE, R. C. : *The design of experiments*, Mimeographed Lecture Notes, University of North Carolina, U.S.A., 1950.
2. CARMICHAEL, R. D. : *Introduction to the theory of groups of finite order*, Ginn, Boston, 1937.
3. COCHRAN, W. G. and COX, M. : *Experimental designs*, John Wiley & Sons, Inc., New York, 1950.
4. FISCHER, R. A. : *The design of experiments*, Oliver & Boyd, Edinburgh, 4th ed., 1947.
5. KEMPTHORNE, O. : *The design and analysis of experiments*, John Wiley & Sons Inc., New York, 1952.
6. MANN, H. B. : *Analysis and design of experiments*, Dover Publications, New York, 1949.
7. YATES, F. : *The design and analysis of factorial experiments*, Imp. Bur. Soil. Sc., Harpenden, England, 1937.


## II. PAPERS


1. ANDERSON, R. L. : Missing-plot techniques, *Biometrics*, 2 (1946).
2. BANERJEE, K. S. : Weighing designs, *Cal. Stat. Assoc. Bull.*, 3 (1950-51).
3. BOSE, R. C. : On the application of Galois fields to the problem of construction of Hyper-Graeco-Latin Squares, *Sankhya*, 3 (1938).
4. BOSE, R. C. and NAIR, K. R. : Partially balanced incopmplete block designs, *Sankhya*, 4 (1939).
5. BOSE, R. C. : Mathematical theory of the symmetrical factorial design, *Sankhya*, 8, (1947).
6. BOSE, R. C. : Mathematics of factorial designs, *Proc. Int. Cong. Math.*, Vol. I (1950).
7. CHASSAN, J. B. : A statistical derivation of a pair of trigonometric inequalities, *Ame. Math. Monthly*, 62 (1955).
8. CHOWLA, S. and RYSER, H. J. : Combinatorial problems, *Can. J. Math.*, 2 (1950).
9. CONNOR, W. S. and CLATWORTHY, W. H. : Some theorems for partially balanced designs, *Ann. Math. Stat.*, 25 (1954).
10. HOTELLING, H. : Some improvements in weighing and other experimental techniques, *Ann. Math. Stat.*, 15 (1944).
11. KISHEN, K. : On fractional replication of the general symmetrical factorial design, *Jour. Ind. Soc. Agr. Stat.*, 1 (1948).
12. KISHEN, K. : Recent Developments in Experimental Design, Presidential address to the Section of Statistics of the 45th Indian Science Congress, Madras, 1958.
13. KSHIRSAGAR, A. M. : A note on the total relative loss of information in any design, *Cal. Stat. Assoc. Bull*, 7 (1957).
14. KSHIRSAGAR, A. M. : A note on incomplete block designs, *Ann. Math. Stat.*, 29 (1958).
15. LYOD, E. H. : Least-squares estimation of location and scale parameters using order statistics, *Biometrika*, 39 (1952).
16. MANN, H. B. and RYSER, H. J. : Systems of distinct representatives, *Ann. Math. Monthly*, 60 (1953).

17. NAIR, K. R. and RAO, C. R. : Confounding in asymetrical factorial experiments, *Jour. Roy. Stat. Soc.*, B, 10 (1948).
18. NANDI, H. K. : On the efficiency of experimental designs, *Cal. Stat. Assoc. Bull.*, 3 (1951).
19. PLACKETT, R. L. and BURMAN, J. P. : The design of optimum multifactorial experiments, *Biometrika*, 33 (1943-46).
20. RAO, C. R. : On the linear combination of observations and general theory of least squares, *Sankhya*, 7 (1946).
21. RAO, C. R. : General Methods of analysis for incomplete block designs, *Jour. Amer. Stat. Assoc.*, 42 (1947).
22. RAO, V. R. : A note on balanced designs, *Ann. Math. Stat.*, 29 (1958).
23. SHRIKHANDE, S. S. : The impossibility of certain symmetrical balanced incomplete block designs, *Ann. Math. Stat.*, 21 (1950).
24. SHRIKHANDE, S. S. : Impossibility of some affine resolvable balanced incomplete block designs, *Sankhya*, 11 (1951).
25. SILLITTO, G. P. : An extension property of a class of balanced incomplete block designs, *Biometrika*, 44 (1957).
26. SMITH, C. A. B. and HARTLEY, H. O. : The construction of Youden Squares, *Jour. Roy. Stat. Soc.* B, 10, (1948).
27. STEVENS, W. L. : The completely orthogonalised square, *Ann. Eng.*, 9 (1938).
28. TOCHER, K. D. : The design and analysis of block experiments, *Jour. Roy. Stat. Soc.*, B, 14 (1952).
29. VARTAK, M. N. : On an application of Kronecker product of matrices to statistical designs, *Ann. Math. Stat.*, 26 (1955).
30. YATES, F. : The analysis of replicated experiments when the field results are incomplete, *Emp. Jour. Exp. Agr.*, 1 (1933).

# INDEX